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The oscillations of a body with an orbital tethered system☆

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Abstract

The motion about a centre of mass of a rigid body with a tethered system, designed to launch a re-entry capsule from a circular orbit is considered. In the deployment of the tethered system the direction and value of the tensile strength of the tether vary and, if the point of application of the tensile strength does not coincide with the centre of mass of the body, a moment occurs which leads to oscillations of the body with variable amplitude and frequency. A non-linear equation of the perturbed motion of the body about the centre of mass under the action of the tensile force of the tether and the gravitational moment is derived. Assuming that the change in the value and direction of the tensile force is slow and also that the gravitational moment is small, approximate and exact solutions of the non-linear differential equation of the unperturbed motion are obtained in terms of elementary functions and elliptic Jacobi functions. For perturbed motion, the action integral is expressed in terms of complete elliptic integrals of the first and second kind.

In the majority of publications devoted to an analysis of space tethered systems, the object of the investigation is the tether and the load, in which the satellite is regarded as a point mass.[1–4] In this paper we assume that the law of variation of the tensile force of the tether and the trajectory of the load, attached to the tether, are known, and we investigate the oscillations of the satellite as a rigid body under the action of the tensile force of the tether and the gravitational moment.

1. Formulation of the problem

Consider the motion of a body (a satellite) about a centre of mass during the dynamic deployment of a tethered system with a re-entry capsule. In dynamic deployment the tether is released more rapidly than in static deployment,[2,3] and, under the action of the Coriolis force, the capsule is deflected from the vertical, and then, after the tether unfolds to its complete length, return motion of the capsule to the vertical begins. The tensile force of the tether, variable in value and direction, produces an additional moment, under the action of which the satellite performs non-stationary oscillations about the centre of mass, which, in turn, leads, for example, to the occurrence of undesirable additional microaccelerations. The gravitational and Coriolis forces, which lie in the orbital plane of the satellite, have a decisive effect on the motion of the tethered system, and hence it is completely justified to consider the plane motion of the tethered system and the body.

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The aim of this paper is to obtain approximate and exact solutions of the equations which describe the perturbed and unperturbed motion about a centre of mass of a body with a tethered system.

2. Perturbed and unperturbed motions

Consider a mechanical system (Fig. 1), consisting of a satellite c with centre of mass at the point O, a tether P_1P_2 and a re-entry capsule P_2. The satellite moves in a circular orbit (represented in Fig. 1 by the dashed curve). We introduce in the orbital plane a system of coordinates Ox_1y_1, in which the Ox_1 axis coincides with the local vertical, and a system of coordinates $Oxyz$ connected with the satellite, in which the Oxy plane coincides with the orbital plane and the Ox axis is directed along the longitudinal axis of the satellite.

To derive the equation describing the motion of the satellite about the centre of mass, we will use the theorem of the change in the angular momentum[5] projected onto the Oz axis, perpendicular to the plane of motion Oxy (Fig. 1). We obtain

$$C\ddot{\alpha} = -T\Delta\sin(\alpha-\varphi)+M_g; \quad M_g = -3n^2(B-A)\sin\alpha\cos\alpha \tag{2.1}$$

Here α is the angle between the longitudinal axis of the body and the local vertical, A, B and C are the principal components of the inertia tensor of the body in the coupled system of coordinates $Oxyz$, T is the value of the tensile force of the tether, φ is the angle between the line of action of the tensile force of the tether and the local variable, $\Delta = OP_1$ (Fig. 1) and M_g is the gravitational moment ($n = \sqrt{\gamma M/r_3^3}$, where γ is the universal gravitational constant, M is the mass of the Earth, and r_3 is the distance from the body to the centre of the Earth).[6]

Suppose the tensile force of the tether and the angle between the line of action of the tensile force of the tether and the vertical are slowly varying functions of the slow time $\tau = \varepsilon t$, where ε is a small parameter. We will consider the case when the gravitational moment is considerably less than the moment produced by the tensile force of the tether, and we will assume that its order is equal to $O(\varepsilon)$, after which Eq. (2.1) takes the form

$$\ddot{\alpha}+\eta(\tau)\sin\alpha-\nu(\tau)\cos\alpha = -\varepsilon\kappa\sin 2\alpha \tag{2.2}$$

where

$$\eta(\tau) = \omega^2(\tau)\cos\varphi(\tau),\ \nu(\tau) = \omega^2(\tau)\sin\varphi(\tau),\ \omega^2(\tau) = T(\tau)\frac{\Delta}{C},\ \varepsilon\kappa = \frac{3}{2}n^2\frac{B-A}{C} \tag{2.3}$$

When $\varepsilon = 0$ we obtain the equation of unperturbed motion, and it allows of a first integral, namely, the energy integral

$$\dot{\alpha}^2/2-\eta\cos\alpha-\nu\sin\alpha = H \tag{2.4}$$

where η, ν, H are certain constants.

We obtain a solution of the equation of unperturbed motion by considering only the oscillatory motion between two positions: $\alpha_1 = \alpha_{min}$ and $\alpha_2 = \alpha_{max}$. We will choose the following initial conditions

$$t = 0:\ \alpha_0 = \alpha_2, \quad \dot{\alpha}_2 = 0$$

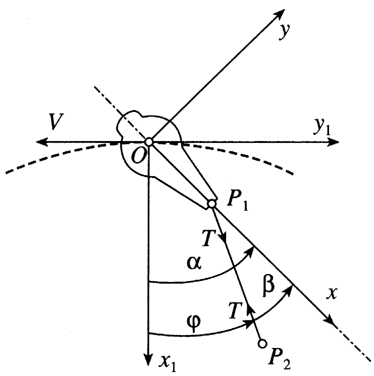

Fig. 1.

We then have

$$H = -\eta\cos\alpha_2 - \nu\sin\alpha_2 \tag{2.5}$$

Separating the variables in Eq. (2.4), we obtain

$$t = -\int_{\alpha_2}^{\alpha}\frac{d\alpha}{\sqrt{f(\alpha)}};\quad f(\alpha) = a + b\sin\alpha + c\cos\alpha,\quad a = 2H,\quad b = 2\nu,\quad c = 2\eta \tag{2.6}$$

Note that $\alpha_1 = \alpha_{min}$ and $\alpha_2 = \alpha_{max}$ are the roots of the equation $f(\alpha) = 0$.

We will use the replacement[7]

$$\alpha = 2\psi + \delta;\quad \delta = \operatorname{arctg}(b/c) = \operatorname{arctg}(\nu/\eta) = \varphi \tag{2.7}$$

(see Eq. (2.3)), in which case integral (2.6) takes the form

$$t = -\frac{2}{\sqrt{a+p}}\int_{\psi_2}^{\psi}\frac{d\psi}{\sqrt{1-q^2\sin^2\psi}};\quad p = \sqrt{b^2+c^2},\quad q^2 = \frac{2p}{a+p} \tag{2.8}$$

Making the replacement

$$\sin\psi = q^{-1}\sin\gamma \tag{2.9}$$

we convert the elliptic integral (2.8) to normal form with a modulus not exceeding unity

$$\omega t = -\int_{\pi/2}^{\gamma}\frac{d\gamma}{\sqrt{1-k^2\sin^2\gamma}};\quad \omega = \sqrt{\frac{p}{2}} = \sqrt{T\frac{\Delta}{C}},\quad k^2 = q^{-2} = \sin^2\frac{\varphi-\alpha_2}{2} \tag{2.10}$$

Introducing the Jacobi amplitude function,[8] we can rewrite equality (2.10) as follows:

$$\gamma = \operatorname{am}(\mathbf{K}(k) - \omega t, k) \tag{2.11}$$

where $\mathbf{K}(k)$ is the complete elliptic integral of the first kind.

3. The action integral and approximate solutions

For a perturbed single-frequency system with slowly varying parameters (2.2), it is of interest to consider the action integral,[9] which, apart from a factor, can be represented in two forms

$$I(\tau) = \int_t^{t+T_\alpha}\dot{\alpha}^2 dt = 2\int_{\alpha_1}^{\alpha_2}\dot{\alpha}d\alpha \tag{3.1}$$

where $T_\alpha = 4\mathbf{K}(k)/\omega$ is the period of the oscillations of the angle α.

The behaviour of the action integral for the perturbed system (2.2) is determined by the average differential equation[9]

$$\dot{I} = -\varepsilon\kappa\int_t^{t+T_\alpha}\dot{\alpha}\sin 2\alpha dt \tag{3.2}$$

The integrand, by virtue of solution (2.11), taking formulae (2.7), (2.9) and (2.10) into account, is an odd periodic function of time t, and hence the right-hand side of Eq. (3.2) vanishes, while the action integral (3.1) retains its value, i.e. it will be the adiabatic invariant of the perturbed motion

$$I = \text{const} \tag{3.3}$$

We will use the second form for the action integral (3.1) and, employing the second formula of (2.6), we can write it in the form

$$I = 2\int_{\alpha_1}^{\alpha_2} \dot{\alpha} d\alpha = 2\int_{\alpha_2}^{\alpha_2} \sqrt{f(\alpha)} d\alpha$$

Taking the replacements of variables (2.7) and (2.9) into account, we obtain

$$I = 16\omega \int_0^{\pi/2} \frac{\cos^2\gamma d\gamma}{\sqrt{1-k^2\sin^2\gamma}} = 16\omega D(k), \quad D(k) = \mathbf{E}(k) - (1-k^2)\mathbf{K}(k) \tag{3.4}$$

where $\mathbf{E}(k)$ is the complete elliptic integral of the second kind.

For the perturbed motion, the value of the tensile force of the tether and its direction are determined by the known slowly varying functions $T = T(\tau)$ and $\varphi = \varphi(\tau)$. It is obvious that the amplitude values of the angle of deflection of the body from the vertical will also change: $\alpha_1 = \alpha_{\min}$ and $\alpha_2 = \alpha_{\max}$. These values are related by the equation

$$H(\alpha_1) = H(\alpha_2) \tag{3.5}$$

The function $H(\alpha)$ is given by formula (2.4).

From relations (3.3) and (3.4) for the perturbed motion we have the following relation between the frequency ω and the modulus k

$$\omega D(k) = \text{const} \tag{3.6}$$

According to the last two relations of (2.10) the frequency ω depends on the known value of the tensile force of the tether $T(\tau)$, while the modulus k depends on the known angle $\varphi(\tau)$ and on the unknown amplitude α_m ($\alpha_1 = \alpha_{\min}$ or $\alpha_2 = \alpha_{\max}$), i.e. formula (3.6) gives the implicit dependence of the amplitude α_m on the known functions.

We use the method previously described in Ref. 10 and obtain from Eq. (3.6) in explicit form an approximate analytical dependence of the amplitude of the oscillation of the body on the slow time τ. After expanding the complete elliptic integrals of the first and second kind in power series,[8] we obtain from (3.6)

$$\frac{h}{\omega} = \frac{\pi}{2}\left(k^2 + \sum_{j=1}^{\infty}\left[\frac{(2j-1)!!}{2^j j!}\right]^2\left(k^2 - \frac{2j}{2j-1}\right)k^{2j}\right) (h = \text{const})$$

After inverting this power series[11] and rewriting the last relation of (2.10), by virtue of Eq. (3.5), in the form

$$k^2 = \sin^2\frac{\varphi - \alpha_2}{2} = \sin^2\frac{\varphi - \alpha_1}{2}$$

we write the following expression for the minimum and maximum angle of deflection of the body from the vertical in the following form

$$\alpha_{1,2} = \varphi(\tau) \mp 2\arcsin\left\{\frac{h}{\omega(\tau)} - \frac{1}{2\pi}\left[\frac{h}{\omega(\tau)}\right]^2 - \frac{1}{4\pi^2}\left[\frac{h}{\omega(\tau)}\right]^3 - \dots\right\}^{1/2} \tag{3.7}$$

4. Approximate solution and microaccelerations

The moment of the tensile force of the tether is equal to zero when the longitudinal axis of the body coincides with the line of action of this force, and oscillations of the body occur with respect to the line of action of the tensile force. To obtain solutions corresponding to the linear formulation, it is more convenient to change to a new variable, namely, the angle between the axis of the body and the line of action of the tensile force: $\beta = \alpha - \varphi$. In this case the perturbed equation of motion for small angles β takes the form

$$\ddot{\beta} + \omega^2(\tau)\beta = -\varepsilon(\ddot{\varphi} + \kappa\sin 2\varphi - 2\beta\kappa\cos 2\varphi) \tag{4.1}$$

The first term in parenthesis is due to the non-uniformity of the deflection of the tether from the local vertical, while the second and third terms are due to the gravitational moment acting on the body.

When $\varepsilon = 0$ system (4.1) reduces to the equation of unperturbed motion.

We change to amplitude (x) - phase (y) variables by replacing the variables

$$\beta = x\cos y, \quad \dot{\beta} = -x\omega\sin y; \quad y = \omega(t+t_0) \tag{4.2}$$

Following the well-known approach,[12] for the perturbed equation (4.1) we will write in terms of the variables x, y the expression for the action integral,[13] which is the adiabatic invariant

$$I = \frac{1}{2\pi}\int_0^{2\pi}\omega x^2\sin^2 y dy = \frac{\omega x^2}{2} = \text{const} \tag{4.3}$$

Note that solution (4.3) can be obtained directly from solution (3.7) for a small angle $\beta = \alpha - \varphi$.

If, at a certain instant of time $t = t_0$, we know the value of the action integral (4.3)

$$I_0 = \omega_0 x_0^2/2$$

where ω_0, x_0 are the values of the frequency and amplitude of the oscillation of the angle β when $\tau = \tau_0$, then, using equality (4.3), we have an analytic expression for the amplitude of the angle of deflection of the longitudinal axis of the body from the line of action of the tensile force of the tether

$$x(\tau) = x_0\sqrt{\omega_0/\omega(\tau)}$$

Hence we have the following expression for the minimum and maximum angle of deflection of the body from the vertical

$$\alpha_{1,2} = \varphi(\tau) \mp x_0\sqrt{\omega_0/\omega(\tau)} \tag{4.4}$$

The oscillatory process which occurs due to deployment of the tethered system gives rise to an additional acceleration $W = \sqrt{W_n^2 + W_\tau^2}$, where $W_n = \dot{\alpha}^2 r$, $W_\tau = \ddot{\alpha} r$ are the normal and tangential accelerations (r is the distance from the point at which the acceleration is determined to the centre of mass of the satellite). Changing to the variable β and then to the amplitude-phase variables, as given by formulae (4.2), we have

$$W = x\omega^2 r\sqrt{x^2\sin^4 y + \cos^2 y}$$

It can be shown that if the amplitude $x \le 1$, the integrand reaches a maximum value, equal to unity, i.e. the maximum value of the overload is given by the formula

$$W_{\max} = x\omega^2 r$$

Substituting the solution for the amplitude (4.4) into this formula, we obtain an analytic expression for the envelope of the curve of the additional overload, due to deployment of the tethered system:

$$W_{\max}(\tau) = x_0 r\omega_0^{1/2}[\omega(\tau)]^{3/2} = C^{-1}\Delta x_0 r T_0^{1/4}[T(\tau)]^{3/4} \tag{4.5}$$

5. Modelling of the motion

We will consider the dynamic deployment of a tethered system from a satellite moving in a circular orbit of radius $r_3 = 6621$ km. In Fig. 2 we show the control law of the tensile force of the tether $T(t)$ (the continuous curve) and the angle of deflection of the tether from the vertical $\varphi(t)$ (the dashed curve). The point for which calculation of the accelerations is carried out is a distance $r = 1$ m from the centre of mass of the satellite.

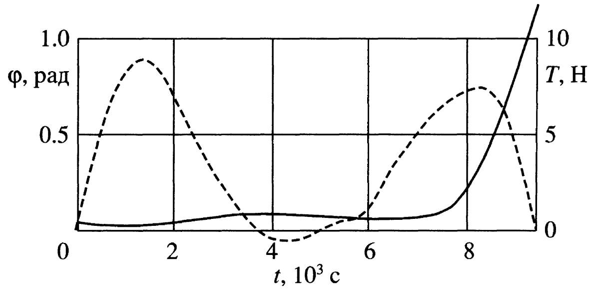

Fig. 2.

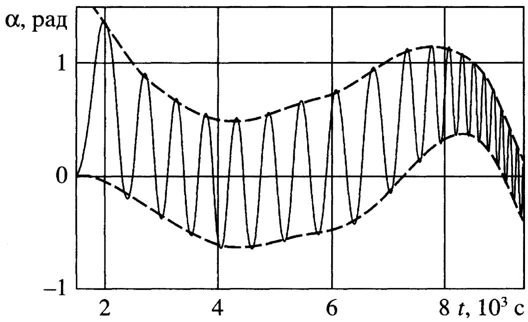

Fig. 3.

To compare the results of the modelling we will choose the following parameters of the satellite

$$A = 10^3 \text{ кг} \cdot \text{м}^2, \quad B = C = 10^4 \text{ кг} \cdot \text{м}^2, \quad \Delta = 2 \text{ м}$$

At the time $t_0 = 1500$ s the satellite is oriented along the vertical $\alpha_0 = \dot{\alpha}_0 = 0$.

In Fig. 3 we show the angle of deflection of the satellite from the vertical, obtained by numerical integration of the perturbed equation (2.1), and the envelopes of this relation, calculated from the approximate formulae (3.7) and (4.4) (the dashed curves). The approximate solutions (3.7) and (4.4) are practically identical (they are indistinguishable on the scale of Fig. 3) and agree extremely well with the results of numerical integration. In the calculations, the angle $\beta = \alpha - \varphi$ is small, which, obviously, also explains the agreement of the results obtained using formulae (3.7) and (4.4).

An analytic estimate of the maximum additional acceleration (4.5) also gives good agreement with the results of numerical modelling (Fig. 4). In the modelling it was assumed that, up to the instant of time $t_0 = 1500$ s, the satellite is oriented along the local vertical ($\alpha = 0$) using the control system, and the control system is then disconnected and

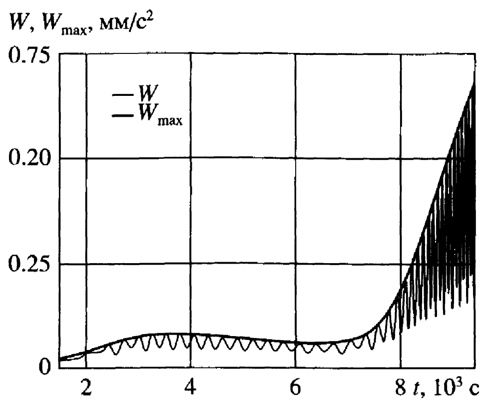

Fig. 4.

it begins to perform oscillations under the action of the tensile force of the tether. The additional acceleration then reaches extremely large values (of the order of 1 m/s^2).

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