Nonlinear Dyn
DOI 10.1007/s11071-011-0225-x

ORIGINAL PAPER

Integrable cases in the dynamics of axial gyrostats and adiabatic invariants

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Received: 14 February 2011 / Accepted: 15 September 2011

Abstract This paper presents the study of axial gyrostats dynamics. The gyrostat is composed of two rigid bodies: an asymmetric platform and an axisymmetric rotor aligned with the platform principal axis. The paper discusses three types of gyrostats: oblate, prolate, and intermediate. Rotation of the rotor relative to the platform provides a source of small internal angular momentum and does not affect the moment of inertia tensor of the gyrostat. The dynamics of gyrostats without external torque is considered. The dynamics is described by using ordinary differential equations with Andoyer–Deprit canonical variables. For undisturbed motion, when the internal moment is equal to zero, the stationary solutions are found, and their stability is studied. General analytical solutions in terms of elliptic functions are also obtained. These results can be interpreted as the development of the classical Euler case for a solid, when to one degree of freedom—the relative rotation of bodies—is added. For disturbed motion of the gyrostats, when there is a system with slowly varying parameters, the adiabatic invariants are obtained in terms of complete elliptic integrals, which are approximately equal to the first integrals of the disturbed system. The adiabatic invariants remain approximately constant along a trajectory for long time intervals during which the parameter changes considerably. The results of the study can be useful for the analysis of dynamics of dual-spin spacecraft and for studying a chaotic behavior of the spacecraft.

Keywords Axial gyrostats · Andoyer–Deprit variables · Adiabatic invariants

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1 Introduction

The dynamics of a rotating body is a classic topic of study in mechanics. In the eighteenth and nineteenth centuries, several aspects of the motion of a rotating rigid body were studied by such famous mathematicians as Euler, Cauchy, Jacobi, Poinsot, Lagrange, and Kovalevskaya. However, the study of the dynamics of rotating bodies is still very important for numerous applications such as the dynamics of satellite-gyrostat, spacecraft, robotics, and the like. For example, Rumyantsev [1] developed Lyapunov's ideas arising from the theory of stability of the equilibrium figure of a rotating liquid contained within a gyrostat. The Lyapunov–Rumyantsev theorem is widely used in the design of artificial satellites and liquid-filled projectiles. Andoyer–Deprit canonical variables are used in Hamiltonian structure of an asymmetric gyrostat in the gravitational field [2]. Kinsey et al. [3] focused upon the capture dynamics of the precession phase lock, a phenomenon that could prevent the despin of

Published online: 07 October 2011

a dual-spin spacecraft by developing a control strategy that employed closed-loop feedback control of the motor torque when the system was near resonance. Hall [4] proposed a procedure based upon the global analysis of the rotational dynamics. Hall and Rand [5] considered spinup dynamics of classical axial gyrostat composed of an asymmetric platform and an axisymmetric rotor. They obtained averaged equations of motion for slowly varying relative rotation of the bodies (disturbed motion) and analytical solutions in terms of Jacobi's elliptic functions for the projections of angular momentum in the case of constant relative rotation (undisturbed motion). Anchev [6] derived necessary conditions of stability of permanent rotations of a heavy gyrostat with arbitrary mass distribution and determined the regions of stability on the conical locus formed by the permanent axes. Spinup problems for axisymmetric gyrostats have been investigated by Kane [7]. Elipe [8] investigated a free gyrostat with three flywheels rotating about the three principal axes of inertia and without any external forces or torques. El-Sabaa [9] used Hamiltonian function of the problem of gyrostat is written in terms of Deprit's transform to obtain periodic solutions and the condition for their stability. Recently, many authors have studied different problems of gyrostats in various situations, most of them related to the dynamics of artificial satellites. Some of these authors have received analytical solutions of the equations of motion of free gyrostats [10] or under the influence of a central field [11]. Cochran et al. [10] extended the previous results for axial gyrostats, obtaining solutions for the Euler angles in terms of elliptic integrals. Cavas and Vigueras [11] obtained solutions for Euler angles in terms of functions of the time. El-Gohary [12–14] studied the control moments sufficient to ensure asymptotic stability of the equilibrium position and rotational motion of a gyrostat, using the Lyapunov function. The problem of optimal stabilization of the rotational motion of a symmetrical rigid body with the help of internal rotors is studied by El-Gohary [15]. The control of the angular motion of a rigid body by means of the rotors is studied in [16]. Tsogas et al. [17] investigated the dynamics of a gyrostat satellite acted upon by the Newtonian forces of N coplanar big bodies, $N-1$ of which are arranged at equal distances on the periphery of a circle, while the Nth body is located at the mass center of the system; they derived the gyrostat's equations of motion and its equilibrium states as well as their stability. Kalvouridis [18] studied the dynamics of a small gyrostat satellite acted upon by the Newtonian forces of two big bodies of equal masses which rotate around their center of mass. Balsas et al. [19] studied two body roto-translatory problems where the rotation of one of them influences strongly in the orbital motion of the system using the canonical action-angle variables. Neishtadt and Pivovarov [20] considered the evolution of the rotation of a gyrostat satellite with slow rotor spinup and worked out formulas for the probabilities which arise due to separatrix crossing. Aslanov [21] obtained explicit analytical time dependences of the Andoyer–Deprit variables corresponding to heteroclinic orbits for all phase portrait forms of undisturbed motion of axial gyrostats, when the internal moment is equal to zero.

Although previous works give insight into the behavior of axial gyrostats, equations of motion have not been reduced to the system with one degree of freedom and exact analytical solutions have not been found for undisturbed motion using Andoyer–Deprit canonical variables. In the disturbed case, the first integrals of averaged equations have not been obtained either. Therefore, this paper presents the study of nonlinear dynamic behavior of the classical axial gyrostats with zero external torque in undisturbed (internal moment is equal to zero) and disturbed (internal moment is small) cases. The classical gyrostat model [22] has balanced axisymmetric rotor coupled to a platform that may be unbalanced. We consider three types of the gyrostats: oblate, prolate, and intermediate.

The aim of this paper is to get stationary solutions, exact general analytical solutions and separatrix trajectories of the undisturbed equations of the motion and to define the first integrals of the disturbed motion of axial gyrostats.

This paper is organized in the following way: in Sect. 1, the aim of the paper is formulated; in Sect. 2, the motion of axial gyrostats as two rigid bodies connected by a rigid shaft is considered. One of the bodies is a rotor, axisymmetric in relation to the axis of rotation, and the other body is an asymmetric platform or core body. There is no external moment, and the angular velocity of the rotor relative to the platform is changing slowly. The gyrostats dynamics is described by using ordinary differential equations in the Andoyer–Deprit canonical with one slowly changing parameter. Section 3 gives the analysis of the undisturbed motion of axial gyrostats in the case when the angular velocity of the rotor relative to the platform is constant. Stationary position gyrostats and their stability are determined according to their moments of inertia. In Sect. 4, a bifurcation diagram and phase portraits are constructed for three types of gyrostats: oblate, prolate, and intermediate. The main features of the phase space of the unperturbed system are defined. In Sect. 5, exact general analytical solutions and separatrix trajectories for the undisturbed motion of the three types of gyrostats are found in terms of Jacobi's elliptic functions and elementary functions. In Sect. 6, adiabatic invariants in terms of complete elliptic integrals are obtained for the disturbed motion gyrostats when there is a system with a slowly varying parameter. These adiabatic invariants are approximately equal to the first integrals of the disturbed system. The adiabatic invariants remain approximately constant along a trajectory for long time intervals during which the parameter changes considerably. By means of computer numerical simulations of perturbed motion, we use a numerical method to check the preservation constancy of the adiabatic invariants.

2 Equations of motion

The gyrostat $(P+R)$ consists of a balanced rotor (R), which is an axisymmetric rigid body, and of an unbalanced platform (P) as shown in Fig. 1. The differential equations of the motion for the angular momentum variables of a rigid axial gyrostat with zero external torque may be written as [8]

$$\frac{dh_1}{dt} = \frac{I_2 - I_3}{I_2 I_3} h_2 h_3, \tag{1}$$

$$\frac{dh_2}{dt} = \left(\frac{I_3 - I_P}{I_3} h_1 - h_a\right)\frac{h_3}{I_P}, \tag{2}$$

$$\frac{dh_3}{dt} = \left(\frac{I_P - I_2}{I_2} h_1 + h_a\right)\frac{h_2}{I_P}, \tag{3}$$

$$\frac{dh_a}{dt} = g_a \tag{4}$$

where e_i are principal axes of $P+R$ $(i = 1, 2, 3)$; g_a is the torque applied by P on R about e_1; $h_a = I_S(\omega_S + \omega_1)$ is the angular momentum of R about e_1; $h_1 = I_1\omega_1 + I_S\omega_S$ is the angular momentum of $P+R$ about e_1; $h_i = I_i\omega_i$ are the angular momentum

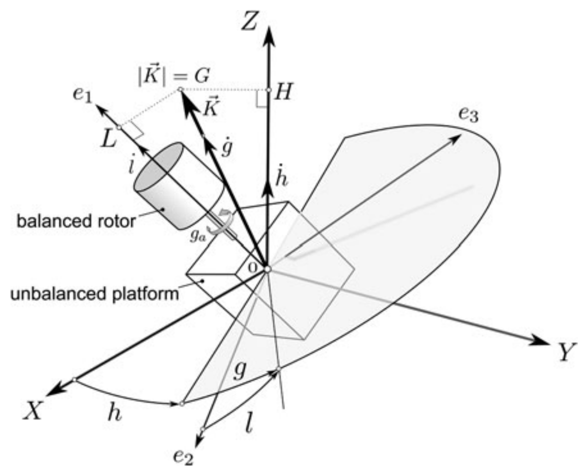

Fig. 1 The axial gyrostat

of $P+R$ about e_i $(i = 2, 3)$; I_i are the moments of inertia of $P+R$ about e_i $(i = 1, 2, 3)$; $I_P = I_1 - I_S$ is the moment of inertia of P about e_1; I_S is the moment of inertia of R about e_1; t is time, ω_i are the angular velocities of P about e_i $(i = 1, 2, 3)$; ω_S is the angular velocity of R about e_1 relative to P.

Since there are external moments, the angular momentum is conserved and the first integral of the motion is

$$G = \sqrt{h_1^2 + h_2^2 + h_3^2} = \text{const.} \tag{5}$$

This first integral can be used to reduce the number of (1)–(3) by one. However, it gives complicated equations of the motion. The equations of the motion can be simplified by using canonical Andoyer–Deprit variables [23, 24]: l, g, h, L, G, H. In our case, the first integral (5) is directly included in the Andoyer–Deprit variables. Using the change of variables

$$h_1 = L, \quad h_2 = \sqrt{G^2 - L^2}\sin l, \\ h_3 = \sqrt{G^2 - L^2}\cos l \tag{6}$$

we obtain the equations of the motion in terms of Andoyer–Deprit variables

$$\dot{l} = \frac{1}{I_P}\left[L - h_a - \frac{1}{2}L(a + b + (b - a)\cos 2l)\right], \tag{7}$$

$$\dot{L} = \frac{1}{2I_P}(b - a)\left(G^2 - L^2\right)\sin 2l, \tag{8}$$

$$\dot{h}_a = g_a \tag{9}$$

where $\dot{x} = dx/dt, a = I_P/I_2, b = I_P/I_3$. The body axes have been chosen so that $I_2 > I_3$ (or equivalently $b > a$).

The transformation of (7)–(9) to a dimensionless form is obtained by scaling two momentum, time, and axial torque, as follows:

$$s = \frac{L}{G}, \quad d = \frac{h_a}{G}, \quad \tau = t\frac{G}{I_p}, \quad \varepsilon = \frac{g_a I_p}{G^2}. \tag{10}$$

The derivatives with respect to τ are denoted by a derivative sign $x' = dx/d\tau$. The change of variables (10) leads to the equivalent set of dimensionless equations:

$$l' = s - d - \frac{s}{2}\big[(a+b) + (b-a)\cos 2l\big], \tag{11}$$

$$s' = \frac{1}{2}(b-a)\big(1-s^2\big)\sin 2l, \tag{12}$$

$$d' = \varepsilon. \tag{13}$$

3 Unperturbed motion and stationary solutions

At $\varepsilon = 0$ the perturbed equations (11)–(13) are reduced to an unperturbed canonical system with one degree of freedom:

$$l' = \frac{\partial H}{\partial s} = s - d - \frac{s}{2}\big[(a+b) + (b-a)\cos 2l\big], \tag{14}$$

$$s' = -\frac{\partial H}{\partial l} = \frac{1}{2}(b-a)\big(1-s^2\big)\sin 2l \tag{15}$$

where $d = \text{const}$; H is Hamiltonian by

$$H(l,s) = \frac{1-s^2}{4}\big[(a+b) + (b-a)\cos 2l\big] + \frac{s^2}{2} - sd = h = \text{const}. \tag{16}$$

Solving the expression (16) with respect to $\cos 2l$, we get an equation of the phase trajectory:

$$\cos 2l = \frac{(a+b-2)s^2 + 4ds + 4h - a - b}{(1-s^2)(b-a)}. \tag{17}$$

Let us define the stationary solutions of (14) and (15). Equating these equations to zero leads to four stationary solutions. The first and the second stationary solutions are respectively described by

$$\cos(2l_*) = 1, \qquad s_* = \frac{d}{1-b}, \tag{18}$$

$$\cos(2l_*) = -1, \qquad s_* = \frac{d}{1-a}. \tag{19}$$

The third and the fourth stationary solutions correspond to the cases when the axis of a rotation gyrostat e_1 coincides with the angular momentum, or takes the opposite direction

$$\cos(2l_*) = \frac{2-a-b-2d}{b-a}, \qquad s_* = 1 \tag{20}$$

$$\cos(2l_*) = \frac{2-a-b+2d}{b-a}, \qquad s_* = -1. \tag{21}$$

When the standard procedure of linearization (14) and (15) is performed in the vicinity of a stationary position $\Delta l = l_* - l$, $\Delta s = s_* - s$, a characteristic equation can be written as

$$\begin{vmatrix} \dfrac{\partial^2 H}{\partial s \partial l} - \lambda & \dfrac{\partial^2 H}{\partial s^2} \\ -\dfrac{\partial^2 H}{\partial l^2} & -\dfrac{\partial^2 H}{\partial l \partial s} - \lambda \end{vmatrix} = 0. \tag{22}$$

This characteristic equation for first stationary solution (18) becomes

$$\lambda^2 - (b-a)(1-b)\big(1-s_*^2\big) = 0.$$

The equilibrium position (18) is obviously stable if

$$b > 1 \quad (I_P > I_3) \tag{23}$$

and unstable if

$$b < 1 \quad (I_P < I_3). \tag{24}$$

For the second stationary solution (19), the characteristic equation (22) can be written as

$$\lambda^2 - (b-a)(a-1)\big(1-s_*^2\big) = 0$$

then the second stationary solution (19) will be stable if

$$a < 1 \quad (I_P < I_2) \tag{25}$$

and unstable for

$$a > 1 \quad (I_P > I_2). \tag{26}$$

Thus, the equilibrium position $l_* = n\pi, n \in \mathbb{Z}$ is stable if the moment of inertia of the platform I_P greater than smaller moments of inertia of the gyrostat I_2. It is unstable if I_P less than I_2. The equilibrium position $l_* = \pi/2 + \pi n, n \in \mathbb{Z}$ is stable if the moment of inertia of the platform is less than the largest of the transverse moments of the gyrostat inertia, and it is unstable if more than that moment of inertia.

For the third and fourth stationary solutions (20) and (21), the characteristic equation (22) can be written as

$$\lambda^2 - (b-a)^2\left(1-\cos^2 2l_*\right) = 0.$$

This equation has only real roots, so the third and fourth stationary solutions (20) and (21) are unstable.

4 Bifurcation diagram

An axial gyrostat is oblate if $I_p > I_2$ or, equivalently, $b > a > 1$; it is prolate if $I_p < I_3$ or if $a < b < 1$, and it is intermediate if $I_3 < I_p < I_2$ or if $b > 1 > a$. We have three areas on the bifurcation diagram which correspond to stationary solutions of (18)–(21) and the conditions that determine the stability or instability of (23)–(26) as shown in Fig. 2:

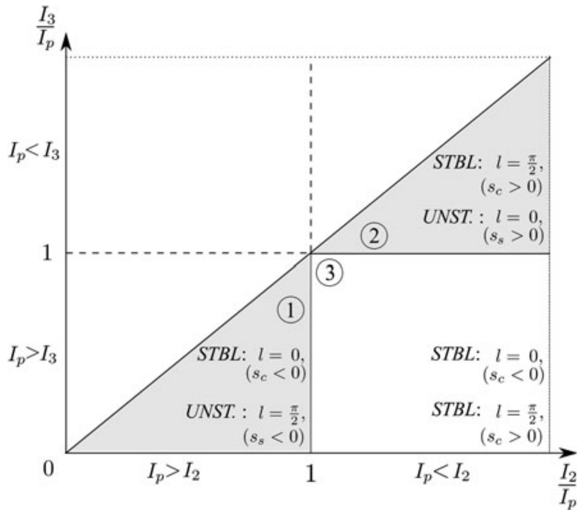

Fig. 2 The bifurcation diagram

(1) oblate gyrostat
centers:

$$l_c = n\pi, \quad n \in \mathbb{Z}; \qquad s_c = d/(1-b) \tag{27}$$

saddles:

$$l_s = \pi/2 + \pi n, \quad n \in \mathbb{Z}; \qquad s_s = d/(1-a) \tag{28}$$

(2) prolate gyrostat
saddles:

$$l_s = n\pi, \quad n \in \mathbb{Z}; \qquad s_s = d/(1-b) \tag{29}$$

centers:

$$l_c = \pi/2 + \pi n, \quad n \in \mathbb{Z}; \qquad s_c = d/(1-a) \tag{30}$$

(3) intermediate gyrostat
centers:

$$l_c = n\pi, \quad n \in \mathbb{Z}; \qquad s_c = d/(1-b) \tag{31}$$

centers

$$l_c = \pi/2 + \pi n, \quad n \in \mathbb{Z}; \qquad s_c = d/(1-a) \tag{32}$$

saddles:

$$l_s = \pm\frac{1}{2}\arccos\frac{2-a-b+2d}{b-a}; \qquad s_s = -1 \tag{33}$$

saddles:

$$l_s = \pm\frac{1}{2}\arccos\frac{2-a-b-2d}{b-a}; \qquad s_s = 1. \tag{34}$$

The examples of phase trajectories for the oblate gyrostat and the prolate gyrostat are shown in Figs. 3 and 4. The phase trajectories are shown in s, l coordinates on plane and, for clarity, on a unit sphere in the spherical coordinates α, l where $\alpha = \arccos(s)$.

In Fig. 5 (for area 3 on the bifurcation diagram), there are two types of separatrix, one of which contains saddles (33) and another—saddles (34). In the phase space bounded by these separatrices, there is continuous motion with a sequential change in the sign of dimensionless momentum s.

5 Integration of the equations of undisturbed motion by quadrature

5.1 Separation of variables

By deleting the coordinate l from (15) and making use of (17), we receive a new form

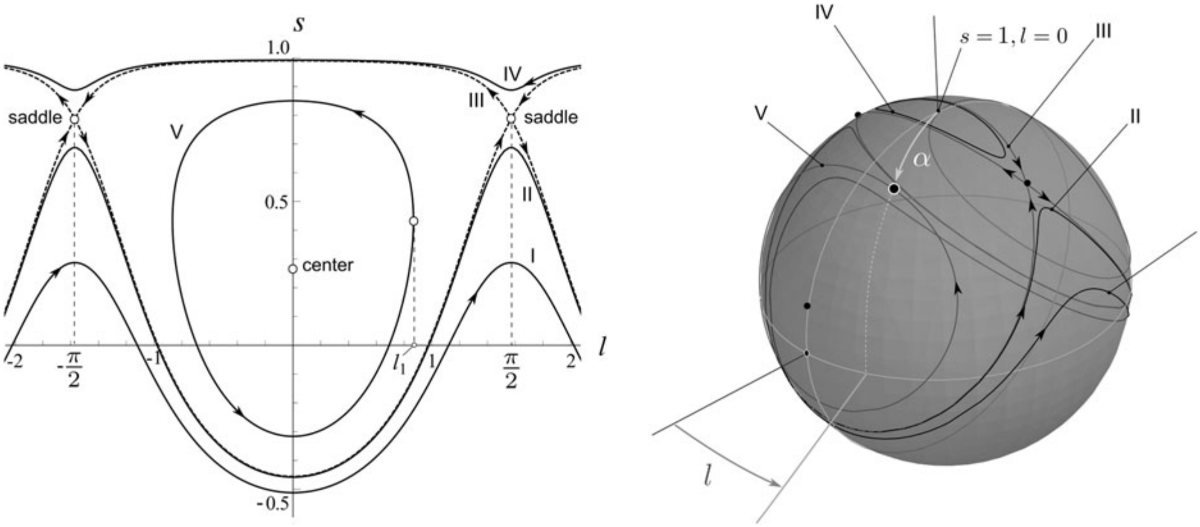

Fig. 3 Phase trajectory for oblate gyrostat: $I_2 = 2.1\ \mathrm{kg\,m^2}$, $I_3 = 1.6\ \mathrm{kg\,m^2}$, $I_p = 2.5\ \mathrm{kg\,m^2}$, $d = -0.15$ ($s_c = 0.267$, $s_s = 0.788$)

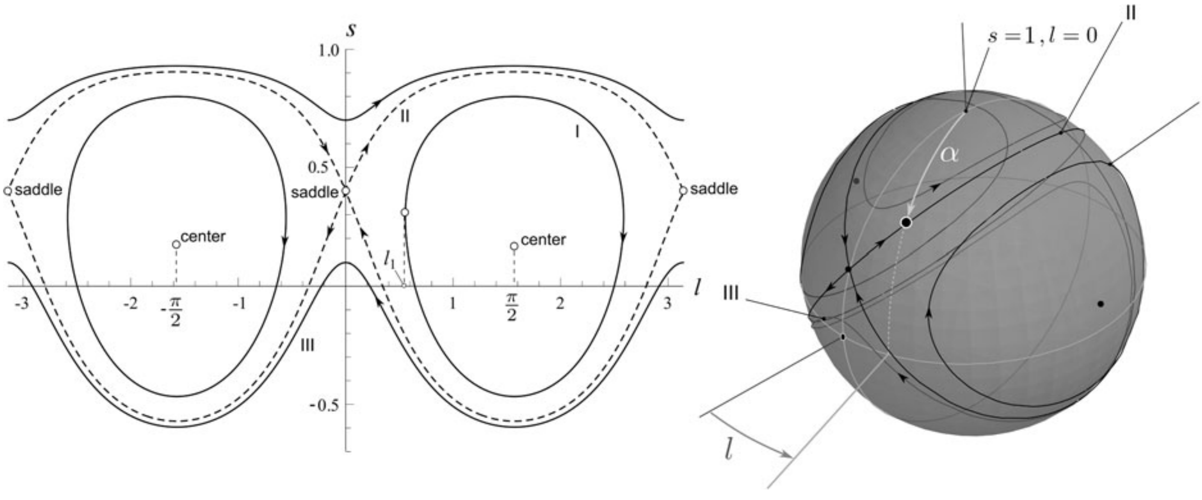

Fig. 4 Phase trajectory for prolate gyrostat: $I_2 = 2.0\ \mathrm{kg\,m^2}$, $I_3 = 1.6\ \mathrm{kg\,m^2}$, $I_p = 1.4\ \mathrm{kg\,m^2}$, $d = 0.05$ ($s_s = 0.4$, $s_c = 0.167$)

$$s' = \pm\frac{1}{2}\Big(\big[(1-s^2)(b-a)\big]^2 - \big[(a+b-2)s^2 + 4ds + 4h - a - b\big]^2\Big)^{1/2} = \pm\sqrt{F(s)} \tag{35}$$

where

$$F(s) = -4 f_a(s) f_b(s) \tag{36}$$

$$f_\gamma(s) = \frac{1}{2}(1-\gamma)s^2 - ds + \frac{\gamma}{2} - h, \quad (\gamma = a, b). \tag{37}$$

Separating the variables in (35) and integrating it, we get

$$\tau = \pm\int \frac{ds}{\sqrt{F(s)}} + \text{const.} \tag{38}$$

In general, this integral is an elliptic integral. Transforming the integral to Legendre normal form [25] depends on the type and location of the roots of the fourth-degree polynomial (36) as the product of two polynomials of the second degree (37). We represent

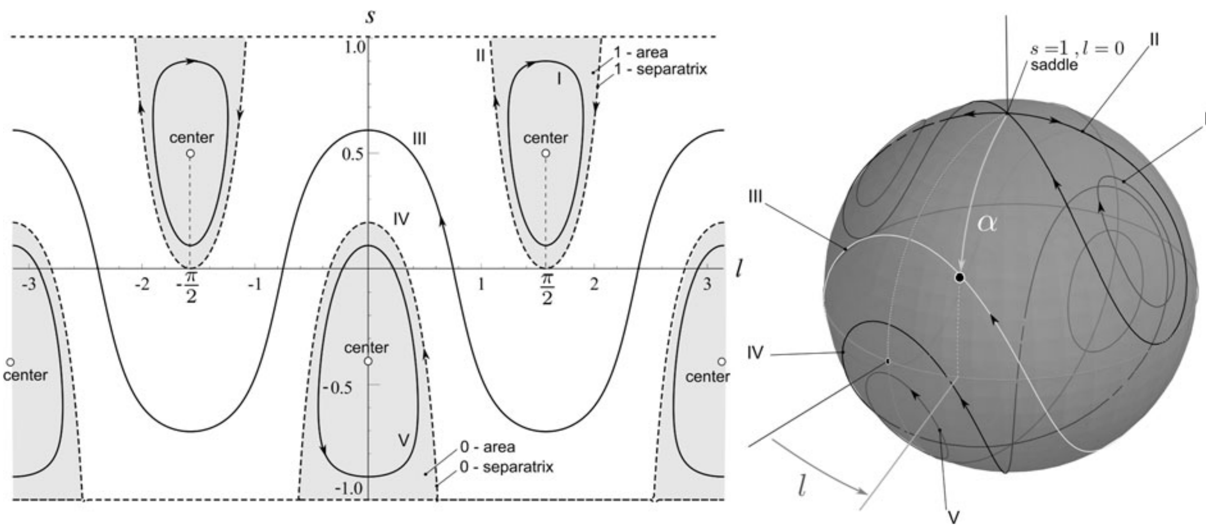

Fig. 5 Phase trajectory for intermediate gyrostat: $I_2 = 2.0\ \mathrm{kg\,m^2}$, $I_3 = 1.6\ \mathrm{kg\,m^2}$, $I_p = 1.8\ \mathrm{kg\,m^2}$, $d = 0.05$ ($s_c = -0.4$ ($l_c = 0$), $s_c = 0.5$ ($l_c = -\pi/2, \pi/2$))

the roots of the quadratic equations

$$f_\gamma(s) = 0 \quad (\gamma = a, b)$$

as

$$s_{1,2}^{\gamma} = \frac{d \pm \sqrt{D_\gamma}}{1-\gamma}, \qquad D_\gamma = d^2 + (2h-\gamma)(1-\gamma). \tag{39}$$

5.2 Analytical solutions for the oblate gyrostat

The type of the roots (39) of the polynomial (36) depends on the value of the constant h. For different types of oblate gyrostat motion ($b > a > 1$), h abide the following condition:

$$h_c > h_L > h_s > h_R \tag{40}$$

where h_L and h_R correspond to libration and rotation, respectively. The constant h in the center (27)—h_c and in the saddle (28)—h_s is

$$h_c = \frac{1}{2}\left(b - \frac{d^2}{1-b}\right), \qquad h_s = \frac{1}{2}\left(a - \frac{d^2}{1-a}\right). \tag{41}$$

First, we consider a motion along separatrices when $h = h_s$. Substituting h in (39) with (41), we derive the following roots:

$$s_{3,4} = s_{1,2}^{a} = \frac{d \pm \sqrt{D_a}}{1-a} = s_s, \quad D_a = 0$$

$$s_{1,2} = s_{1,2}^{b} = \frac{d \mp \sqrt{D_b}}{1-b},$$

$$D_b = \frac{(b-a)[(a-1)(b-1)-d^2]}{a-1} > 0.$$

These real roots are located in the following order:

$$-1 < s_2 < s_3 = s_4 < 0 < s_1 < 1.$$

Consequently, the integral (38) is

$$\lambda\tau = \pm \int \frac{ds}{(s-s_s)\sqrt{(s_1-s)(s-s_2)}} + \text{const} \tag{42}$$

where

$$\lambda = \sqrt{(I_2 - I_p)(I_3 - I_p)/(I_2 I_3)}.$$

Substituting the variables $x = s - s_s$, we present this integral to a well known form [26]

$$\lambda\tau = \pm \int \frac{dx}{x\sqrt{R(x)}} + \text{const}$$
$$= \frac{1}{\sqrt{\alpha}} \ln \frac{2\alpha + \beta x + 2\sqrt{\alpha R(x)}}{x} + \text{const}$$

where

$$R(x) = \alpha + \beta x - x^2$$
$$[\alpha = -(s_1 - s_s)(s_2 - s_s), \beta = s_1 + s_2 - 2s_s].$$

The general solutions for dimensionless momentum can be written as

$$s(\tau) = s_s + \frac{4\gamma_i \alpha \exp(\lambda\sqrt{\alpha}\tau)}{[\gamma_i \exp(\lambda\sqrt{\alpha}\tau) - \beta]^2 + 4\alpha}, \qquad (43)$$
$$\gamma_i = \beta + 2\alpha/(s_i - s_s)$$

where $i = 1$ for an upper separatrix, $i = 2$ for a lower separatrix. The coordinate $l(\tau)$ can be received by using (41) in the formula for the phase trajectory (17). The solutions (43) are found by Aslanov [21].

We have libration's solution if an arbitrary constant $h = h_L$ satisfy the condition (40), and then the phase trajectory belongs to the closed area (Fig. 3) which includes the center (27). The roots of the polynomial (36) with (39) are given by

$$s_{1,2} = s^b_{1,2} = \frac{d \mp \sqrt{D_b}}{1-b},$$
$$D_b = d^2 + (2h_L - b)(1-b) > 0$$
$$s_{3,4} = s^a_{1,2} = s_s \pm i s_k, \qquad s_k = \frac{\sqrt{-D_a}}{1-a},$$
$$D_a = d^2 + (2h_L - a)(1-a) < 0.$$

The integral (38) for two real roots $s_1 > s_2$ and two complex conjugate roots $s_{3,4} = s_s \pm i s_k$ can be written as

$$\lambda\tau = \int_{s_2}^{s} \frac{ds}{\sqrt{(s_1 - s)(s - s_2)(s - s_3)(s - s_4)}}. \qquad (44)$$

Change of variable [25]

$$\left(\tan\frac{\varphi}{2}\right)^2 = \frac{\cos\theta_1}{\cos\theta_2}\frac{s_1 - s}{s - s_2} \qquad (45)$$

converts the integral (44) to Legendre normal form

$$\omega\tau = \int_{\pi}^{\varphi} \frac{d\varphi}{\sqrt{1 - k^2\sin^2\varphi}}$$

where

$$\tan\theta_1 = \frac{s_1 - s_s}{s_k}, \qquad \tan\theta_2 = \frac{s_2 - s_s}{s_k}$$

(θ_1, θ_2 are acute angles),

$$\omega = \frac{\lambda}{\mu}, \qquad k = \frac{\sin\theta_1 - \sin\theta_2}{2},$$

$$\mu = -\frac{(\cos\theta_1\cos\theta_2)^{1/2}}{s_k}$$

We proceed to study of the rotation when $h = h_R$ in the condition (40). The four real roots of the equation $F(s) = 0$ comprise two roots ($s_2 < s < s_1$) which correspond to the upper phase trajectories and two roots ($s_4 < s < s_3$) which refer to the lower phase trajectories as shown in Fig. 3.

$$s_{3,2} = s^a_{1,2} = \frac{d \pm \sqrt{D_a}}{1-a},$$
$$D_a = d^2 + (2h_R - a)(1-a) > 0,$$

$$s_{4,1} = s^b_{1,2} = \frac{d \pm \sqrt{D_b}}{1-b},$$
$$D_b = d^2 + (2h_R - b)(1-b) > 0.$$

Since $D_a < D_b$, the real roots are as follows:

$$-1 < s_4 < s_3 < s_2 < s_1 < 1. \qquad (46)$$

In this case, the integral (38) has the form

$$\lambda\tau = \int_{s_i}^{s} \frac{ds}{\sqrt{(s_1 - s)(s - s_2)(s - s_3)(s - s_4)}} \qquad (47)$$

where the index of the lower limit of the integral $i = 2$, for the upper phase trajectories, and, $i = 4$, for the lower phase trajectories. By a change of variables, [25] the integral (47) can be reduced to Legendre normal integral

$$\omega\tau = \int_0^{\varphi} \frac{d\varphi}{\sqrt{1 - k^2\sin^2\varphi}}. \qquad (48)$$

Then the general solutions for the upper area ($s_2 < s < s_1$) can be written as

$$s = \frac{s_2 s_{31} - s_3 s_{21} sn^2(\omega\tau, k)}{s_{31} - s_{21} sn^2(\omega\tau, k)} \qquad (49)$$

and for the low area ($s_4 < s < s_3$)

$$s = \frac{s_4 s_{31} + s_1 s_{43} sn^2(\omega\tau, k)}{s_{31} + s_{41} sn^2(\omega\tau, k)} \qquad (50)$$

where $sn(\omega\tau, k)$ is elliptic sine

$$\omega = \frac{\lambda}{\mu}, \qquad k^2 = \frac{(s_3 - s_4)(s_2 - s_1)}{(s_3 - s_1)(s_2 - s_4)},$$
$$\mu = 2(s_{31}s_{24})^{-1/2}, \qquad s_{ij} = s_j - s_i.$$

5.3 Analytical solutions for the prolate gyrostat

The saddles and the centers are located at the points indicated in (29), (30) for the prolate gyrostat ($a < b < 1$) and the constant h satisfies the following condition for different types of motion:

$$0 < h_c < h_L < h_s < h_R \tag{51}$$

where $h_c = \frac{1}{2}(a - \frac{d^2}{1-a})$

$$h_s = \frac{1}{2}\left(b - \frac{d^2}{1-b}\right). \tag{52}$$

We write the roots (39) for the separatrices passing through the saddles (30), using the formula (52)

$$s_{3,4} = s^a_{1,2} = \frac{d \pm \sqrt{D_b}}{1-a} = s_s, \quad D_b = 0 \tag{53}$$

$$s_{1,2} = s^b_{1,2} = \frac{d \mp \sqrt{D_a}}{1-a},$$
$$D_a = \frac{(b-a)[(1-a)(1-b) - d^2]}{1-b} > 0. \tag{54}$$

The discriminant D_a is positive and there are real roots

$$-1 < s_2 < s_3 = s_4 < 0 < s_1 < 1.$$

Consequently, the integral (38) can be written as (42), and the solution has the formula (43) which takes into account the roots (53) and (54).

There is a libration when arbitrary constant $h = h_L$ satisfies to condition (51). The roots (39) of the polynomial (36) are

$$s_{1,2} = s^a_{1,2} = \frac{d \pm \sqrt{D_a}}{1-a},$$
$$D_a = d^2 + (2h_L - a)(1-a) > 0$$
$$s_{3,4} = s^b_{1,2} = s_s \pm i s_k,$$
$$s_k = \frac{\sqrt{-D_b}}{1-b}, \quad D_b = d^2 + (2h_L - b)(1-b) < 0.$$

From this, it is clear that the desired solutions coincide with the solutions (45).

In the case of the prolate gyrostat rotation, there are four real roots (39):

$$s_{1,4} = s^a_{1,2} = \frac{d \pm \sqrt{D_a}}{1-a},$$
$$D_a = d^2 + (2h_R - a)(1-a) > 0$$

$$s_{2,3} = s^b_{1,2} = \frac{d \pm \sqrt{D_b}}{1-b},$$
$$D_b = d^2 + (2h_R - b)(1-b) > 0.$$

The location of these real roots

$$-1 < s_4 < s_3 < s_2 < s_1 < 1$$

coincide with the location of the roots (46). Therefore, in this case the general solutions are the solutions (49) and (50).

5.4 Analytical solutions for intermediate gyrostat

The moments of inertia of the intermediate gyrostat are determined by the following relation: $I_3 < I_p < I_2$, ($b < 1 < a$). In this case, we have two groups of areas of librations when the phase trajectories are closed: 0-areas which include centers (31), and 1-areas containing centers (32). These areas correspond to the values of the arbitrary constant of the Hamiltonian h_{L0} and h_{L1}. As shown in Fig. 5, the phase portrait has a single area of rotations and opened trajectories in which $h = h_R$. The constant h for the different types of motion corresponds to the following condition:

$$0 < h_{c1} < h_{L1} < h_{s1} < h_R < h_{s0} < h_{L0} < h_{c0} \tag{55}$$

where h_{c0} and h_{s0} refer to the centers (31) and the saddles (33), respectively

$$h_{c0} = \frac{1}{2}\left(b - \frac{d^2}{1-b}\right),$$
$$h_{s0} = \frac{1}{2} + d \tag{56}$$

h_{c1} and h_{s1} correspond to the centers (32) and the saddles (34)

$$h_{c1} = \frac{1}{2}\left(a - \frac{d^2}{1-a}\right),$$
$$h_{s1} = \frac{1}{2} - d. \tag{57}$$

The constant (56) corresponds to the 0-separatrices (Fig. 5), and the roots (39) can be written as follows:

$$s_{2,4} = s^a_{1,2} = \frac{d \pm \sqrt{D_a}}{1-a}, \quad D_a = [d - (a-1)]^2 > 0,$$

$$s_{1,3} = s^b_{1,2} = \frac{d \pm \sqrt{D_b}}{1-b}, \quad D_b = [(b-1)-d]^2 > 0,$$

All the roots of the polynomial $F(s)$ (36) are real

$$s_1 > 1 > s_2 > 0, \quad s_3 = s_4 = s_{s0} = -1.$$

Consequently, the integral (38) is given by

$$\lambda\tau = \pm \int \frac{ds}{(s+1)\sqrt{(s-s_1)(s-s_2)}} + \text{const}$$

where $\lambda = \sqrt{(A_\Sigma - C)(C - B_\Sigma)/(A_\Sigma B_\Sigma)}$.

With the change of variable, $x = s+1$ this integral leads to a known integral [26]

$$\lambda\tau = \pm \int \frac{dx}{x\sqrt{R(x)}} + \text{const} = \frac{1}{\sqrt{\alpha}} \ln \frac{2\alpha + \beta x + 2\sqrt{\alpha R(x)}}{x} + \text{const}$$

where $R(x) = \alpha + \beta x + x^2$, $[\alpha = (s_1+1)(s_2+1), \beta = -(s_1+s_2+2)]$.

The general solution for dimensionless momentum is

$$s(\tau) = s_s + \frac{4\gamma\alpha\exp(\nu\tau)}{[\gamma\exp(\nu\tau) - \beta]^2 - 4\alpha} \tag{58}$$

where $\nu = \lambda\sqrt{\alpha}$, $\gamma = \beta + 2\alpha/(s_2+1)$.

We substitute the constant (57) in the formulas (39) and get the roots for the 1-separatrices (Fig. 5) in the following way:

$$s_{3,2} = s^a_{1,2} = \frac{d \pm \sqrt{D_a}}{1-a}, \quad D_a = [(1-a)-d]^2 > 0,$$

$$s_{1,4} = s^b_{1,2} = \frac{d \pm \sqrt{D_b}}{1-b}, \quad D_b = [d-(1-b)]^2 > 0.$$

All the roots of the polynomial $F(s)$ are real

$$s_1 < -1 < s_2 < 0, \qquad s_3 = s_4 = s_{s1} = 1.$$

For these roots, the solution (58) is also correct.

For the librations in the 0-areas ($h = h_{L0}$) which include centers (31), we have the following roots of the polynomial (36)

$$s_{1,4} = s^a_{1,2} = \frac{d \pm \sqrt{D_{a0}}}{1-a},$$
$$D_{a0} = d^2 + (2h_{L0} - a)(1-a) > 0, \tag{59}$$

$$s_{3,2} = s^b_{1,2} = \frac{d \pm \sqrt{D_{b0}}}{1-b},$$
$$D_{b0} = d^2 + (2h_{L0} - b)(1-b) > 0. \tag{60}$$

For the 1-areas ($h = h_{L1}$) which includes centers (33), the roots of the polynomial (36) are

$$s_{4,3} = s^a_{1,2} = \frac{d \pm \sqrt{D_{a1}}}{1-a},$$
$$D_{a1} = d^2 + (2h_{L1} - a)(1-a) > 0, \tag{61}$$
$$s_{2,1} = s^b_{1,2} = \frac{d \pm \sqrt{D_{b1}}}{1-b},$$
$$D_{b1} = d^2 + (2h_{L1} - b)(1-b) > 0. \tag{62}$$

The numbering of the roots of (59)–(62) corresponds to the following sequence:

$$s_4 < -1 < s_3 < s_2 < 1 < s_1. \tag{63}$$

The physical motion is realized in the range $s \in (s_3, s_2)$. In this case, the integral (38) becomes

$$\lambda\tau = \int_{s_3}^{s} \frac{ds}{\sqrt{(s-s_1)(s-s_2)(s-s_3)(s-s_4)}} \tag{64}$$

where $\lambda = \sqrt{(A_\Sigma - C)(C - B_\Sigma)/(A_\Sigma B_\Sigma)}$. The elliptic integral (64) reduces to Legendre normal form (48) with the following change of variables [25]:

$$s = \frac{s_3 s_{42} - s_4 s_{32}\sin^2\varphi}{s_{42} - s_{32}\sin^2\varphi}$$

where

$$\omega = \frac{\lambda}{\mu}, \qquad k^2 = \frac{(s_1-s_4)(s_2-s_3)}{(s_1-s_3)(s_2-s_4)},$$
$$\mu = 2(s_{31}s_{42})^{-1/2}, \qquad s_{ij} = s_j - s_i.$$

Then the general solutions can be written as

$$s = \frac{s_3 s_{42} - s_4 s_{32} sn^2(\omega\tau, k)}{s_{42} - s_{32} sn^2(\omega\tau, k)}. \tag{65}$$

Let us consider the area of rotation (Fig. 5) bounded by 0- and 1-separatrices. The range of the arbitrary constant variation $h_R \in (h_{s1}, h_{s0})$ or, according to (56) and (57)

$$h_R \in \left(\frac{1}{2} - d, \frac{1}{2} + d\right).$$

Then the four roots (39) have the form

$$s_{4,3} = s^a_{1,2} = \frac{d \pm \sqrt{D_a}}{1-a},$$
$$D_a = d^2 + (2h_R - a)(1-a) > 0, \tag{66}$$
$$s_{2,1} = s^b_{1,2} = \frac{d \pm \sqrt{D_b}}{1-b},$$
$$D_b = d^2 + (2h_R - b)(1-b) > 0. \tag{67}$$

The physical motion is realized in the range of $s \in (s_3, s_2)$. The location of the roots (66) and (67) corresponds to (63); therefore, the solution (65) describes also the rotation of the intermediate gyrostat.

6 Perturbed motion and adiabatic invariants

6.1 Adiabatic invariant

Suppose that $\varepsilon \neq 0$ is a small parameter, then the set equations of perturbed motion (11)–(13) is a system with one degree of freedom with a slowly varying parameter $d(\tau)$. We have a slow deformation of the Hamiltonian (16)

$$H(l, s, d(\tau)) = \frac{1-s^2}{4}[(a+b) + (b-a)\cos 2l] + \frac{s^2}{2} - s d(\tau) = h(\tau). \tag{68}$$

In this case, we can introduce a new canonical variable I designed to replace as a constant momentum. The so-called action variable I is defined as [27]

$$I(\tau) = \oint s(l)\,dl \tag{69}$$

where the integration is to be carry over a complete period of libration or of rotation, as the case may be. Two types of periodic motion may be distinguished. In the first type, the orbit $s(l)$ is closed, as shown in Figs. 3, 4 and 5. Both l and s are then periodic functions of nondimensional time τ. It is often designated libration. In the second type of periodic motion, the orbit in phase space is such that s is periodic function of l, with period π, as illustrated in Figs. 3, 4 and 5. Equivalently, this kind if motion implies that when l is increased by π, the configuration of system remains essentially unchanged. The action I is the area enclosed by the orbit in phase space. If the parameter $d(\tau)$ changes in time, the system in general does not have any, even approximate, first integral. However, if the parameter $d(\tau)$ is changing slowly, such an approximate first integral exists. This approximate first integral is the action canonical variable (69)

$$I(d(\tau), h(\tau)) \approx \text{const.} \tag{70}$$

The action integral I is then called an adiabatic invariant. It is a function of phase variables and parameter such that its value along a trajectory remains approximately constant on long time intervals on which the parameter changes considerably. Formula (70) allows us to find the Hamiltonian as an implicit function of nondimensional time τ.

Let us solve (68) relatively to s and substitute in the action integral (69)

$$I = \oint \Big(\big(2d \pm \big(4d^2 + [2-(a+b)-(b-a)\cos 2l] \times [4h-(a+b)-(b-a)\cos 2l]\big)^{1/2}\big) \Big/ \big(2-(a+b)-(b-a)\cos 2l\big)\Big) dl. \tag{71}$$

Change of variable $x = \cos 2l$ reduces this integral to the following form:

$$I = -\frac{1}{2}\oint \frac{2d \pm (b-a)\sqrt{(x-x_1)(x-x_2)}}{(b-a)(x-x_3)\sqrt{1-x^2}}\,dx \tag{72}$$

where

$$x_{1,2} = \frac{(1+2h-a-b) \pm \sqrt{D}}{b-a},$$
$$D = (1-2h)^2 - 4d^2 \tag{73}$$

$$x_3 = \frac{2-(a+b)}{b-a}. \tag{74}$$

Integral (71) belongs to a class of elliptic integrals and can be calculated in terms of complete elliptic integrals and integrals of elementary functions.

6.2 Oblate gyrostat

First, we consider a perturbed motion within a closed area, corresponding to libration of the oblate gyrostat $I_p > I_2$ $(b > a > 1)$. The discriminant (73) is always positive in the range $h \in (h_s, h_c)$ referring to this type of motion (see (40)), where $h_c = [b - d^2/(1-b)]/2$ and $h_s = [a - d^2/(1-a)]/2$ according to (41). The discriminant is equal to zero in the points $h_{1,2} = (1 \pm 2d)/2$. Root x_1 corresponds to the point $l_1 = \arccos(x_1)/2$ (Fig. 3), which can be found from the condition

$$\frac{dl(s)}{ds} = 0 \tag{75}$$

where $l(s)$ is the phase trajectory (17). Equation (75) has the following roots:

$$s_{1,2} = \frac{1 - 2h \pm \sqrt{D}}{2d},$$
$$\cos 2l_{1,2} = x_{1,2} = \frac{(1 + 2h - a - b) \pm \sqrt{D}}{b - a}, \tag{76}$$
$$D = (1 - 2h)^2 - 4d^2.$$

It should be noted that the roots of (74) and (76) coincide. The numbering of the roots of (74) will be chosen so that $-1 < x_1 < 1$, and at $h \to h_s$ the point $x_1 \to x_S$.

The action integral (72) defines the area bounded by a closed phase trajectory (Fig. 6), and can be written as

$$I_L = 2\int_{x_1}^{1} \frac{\sqrt{(x - x_1)(x - x_2)}}{(x - x_3)\sqrt{1 - x^2}}\, dx. \tag{77}$$

The change of the variable

$$x = \frac{2x_1 + (1 - x_1)\sin^2\varphi}{2 - (1 - x_1)\sin^2\varphi}$$

leads to the integral (77) to complete elliptic integrals of the first $K(m)$ and the third kind $\Pi(n, m)$ [28]

$$I_L^{(O)} = -\frac{2\sqrt{2}(1 + x_1)}{(1 + x_3)\sqrt{x_1 - x_2}}[(1 + x_2)K(m) - (1 + x_3)\Pi(n_1, m) + (x_3 - x_2)\Pi(n_2, m)]$$

where

$$m = \frac{(1 + x_2)(x_1 - 1)}{2(x_1 - x_2)}, \qquad n_1 = \frac{1 - x_1}{2},$$
$$n_2 = -\frac{(1 - x_1)(1 + x_3)}{2(x_1 - x_3)}$$
$$K(m) = \int_0^{\pi/2} \left(1 - m\sin^2\varphi\right)^{-1/2} d\varphi,$$
$$\Pi(n, m) = \int_0^{\pi/2} \left(1 - n\sin^2\varphi\right)^{-1} \left(1 - m\sin^2\varphi\right)^{-1/2} d\varphi.$$

The motion is observed in the libration area as long as the roots (73) are real and $D = (1 - 2h)^2 - 4d^2 > 0$. If this condition is violated, then the phase trajectory goes into the rotational area (Fig. 6), and then the roots (73) become complex-conjugates

$$x_{1,2} = x_r \pm i x_m, \qquad x_r = \frac{1 + 2h - (a + b)}{b - a},$$
$$x_m = \frac{\sqrt{-D}}{b - a}, \qquad D = (1 - 2h)^2 - 4d^2 < 0. \tag{78}$$

In the case of rotation, the action integral (72) is given by

$$I_R = \int_{-1}^{1} \frac{2d \pm (b - a)\sqrt{(x - x_r)^2 + x_m^2}}{(b - a)(x - x_3)\sqrt{1 - x^2}}\, dx. \tag{79}$$

The change of the variable

$$x = \frac{\nu - \cos\varphi}{1 - \nu\cos\varphi}\quad \nu = \left(\tan\frac{\theta_2 - \theta_1}{2}\right)\left(\tan\frac{\theta_1 + \theta_2}{2}\right),$$
$$\tan\theta_1 = \frac{1 - x_r}{x_m}, \qquad \tan\theta_2 = \frac{-1 - x_r}{x_m},$$

θ_1, θ_2- are acute angles, reduces the integral (79) to complete elliptic integrals of the first and the third kind

$$I_R^{(O)} = \frac{2d\pi}{(b - a)\sqrt{x_3^2 - 1}} \pm 2\mu\left[\frac{(x_r^2 + x_m^2)\nu^2 - 2x_r\nu + 1}{(x_3\nu - 1)\nu}K(m) + \sqrt{\frac{1}{1 - m}}(\nu^2 - 1)\left(\frac{1}{\nu}\Pi(n_1, m_3) - \frac{(x_r - x_3)^2 + x_m^2}{(x_3 - \nu)(x_3\nu - 1)}\Pi(n_2, m_3)\right)\right]$$

where

$$\mu = \frac{(\cos\theta_1\cos\theta_2)^{-1/2}}{x_m}, \qquad m = \sin^2\frac{\theta_1 - \theta_2}{2},$$
$$m_3 = \frac{m}{m - 1}, \qquad n_1 = \nu^2, \qquad n_2 = \left(\frac{\nu x_3 - 1}{\nu - x_3}\right)^2.$$

6.3 Prolate gyrostat

We now consider the action integral of the prolate gyrostat $I_p < I_3$ $(a < b < 1)$ for the libration. In this

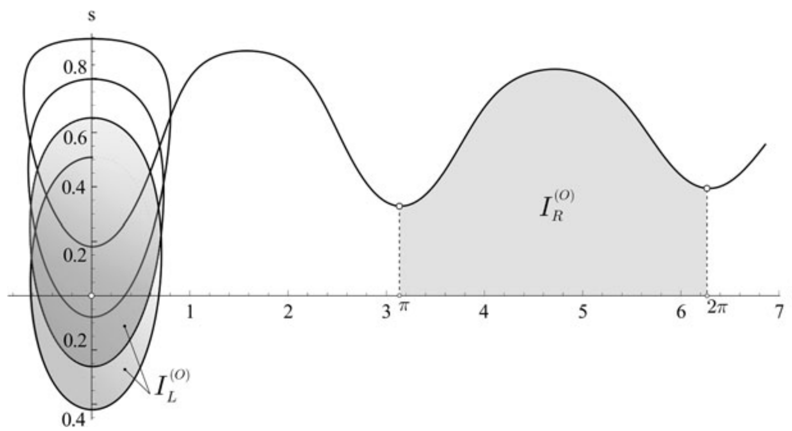

Fig. 6 Phase trajectory for oblate gyrostat with parameters: $I_2 = 2.1\ \mathrm{kg\,m^2}$, $I_3 = 1.6\ \mathrm{kg\,m^2}$, $I_p = 2.5\ \mathrm{kg\,m^2}$, $\varepsilon = -0.005$ the initial conditions of the following: $l_0 = 0$, $s_0 = 0.4$, $d_0 = 0$

case, the centers are located at the points $l_c = \pi/2 + \pi n, n \in \mathbb{Z}$; $s_c = d/(1-a)$, and the saddles are located at the points $l_s = \pi n, n \in \mathbb{Z}$; $s_s = d/(1-b)$, as shown in Fig. 4. We will choose the following location of the roots (73):

$$-1 < x_1 < 1 < x_2 > 1.$$

When we have the liberation of the prolate gyrostat, transformation, similar to (77), gives the following formula for the integral action:

$$I_L^{(P)} = \frac{2\sqrt{2}(x_2+1)}{(x_3+1)\sqrt{x_2-x_1}} \times \left[(x_3+1)\Pi(n_1,m) + (x_1-x_3)\Pi(n_2,m)\right]$$

where

$$m = \frac{(x_1+1)(x_2-1)}{2(x_2-x_1)}, \qquad n_1 = \frac{x_1+1}{x_1-x_2},$$
$$n_2 = \frac{(x_1+1)(x_2-x_3)}{2(x_2-x_1)(x_3+1)}.$$

For the rotation of the prolate gyrostat, roots (73) are related by the condition $1 < x_1 < x_2$, the action integral has the form

$$I_R^{(P)} = \frac{2d\pi}{(b-a)\sqrt{x_3^2-1}} \pm \frac{2(x_2+1)}{(x_3+1)\sqrt{x_2-1}\sqrt{x_1+1}} \times [(x_3+1)\Pi(n_1,m) + (x_1-x_3)\Pi(n_2,m)]$$

where

$$m = \frac{2(x_2-x_1)}{(x_1+1)(x_2-1)}, \qquad n_1 = -\frac{2}{x_2-1},$$
$$n_2 = \frac{2(x_2-x_3)}{(x_2-1)(x_3+1)}.$$

6.4 Intermediate gyrostat

The intermediate gyrostat $I_3 < I_p < I_2$ $(b > 1 > a)$ differs from the oblate and prolate gyrostats by the presence of two types of centers (31), (32), and two types of saddles (33) and (34).

The real roots (73), corresponding to the libration near the center (31), satisfy

$$0 < x_1 < x_2 < (2-a-b+2d)/(b-a).$$

The integral action in the case of libration in the vicinity of the center (31) can be written as

$$I_L^{(I,0)} = \frac{2(x_2-x_1)}{\sqrt{(1-x_1)(1+x_2)}}[\Pi(n_1,m) - \Pi(n_2,m)]$$

where

$$m = \frac{(x_1+1)(x_2-1)}{(x_1-1)(1+x_2)}, \qquad n_1 = \frac{x_2-1}{x_1-1},$$
$$n_2 = \frac{(x_2-1)(x_1-x_3)}{2(x_1-1)(x_2-x_3)}.$$

The real roots (73), corresponding to the libration near the center (32), satisfy

$$-1 < x_1 < x_2 < (2-a-b-2d)/(b-a).$$

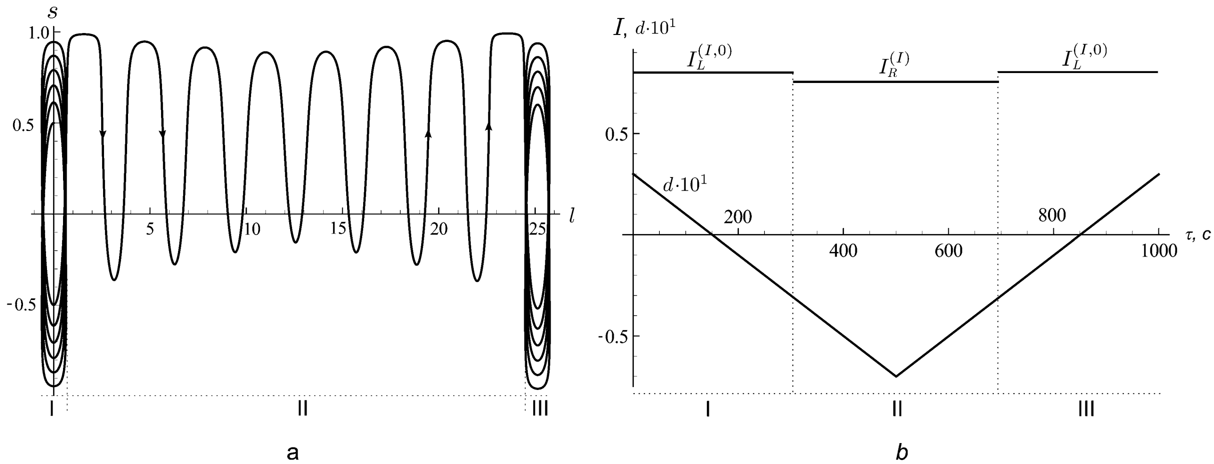

Fig. 7 Disturbed motion of intermediate gyrostat (**a**) the phase portrait $s = s(l)$ of the disturbed motion trajectory for the intermediate gyrostat: $I_2 = 2.0\ \mathrm{kg\,m^2}$, $I_3 = 1.6\ \mathrm{kg\,m^2}$, $I_p = 1.8\ \mathrm{kg\,m^2}$; (**b**) the action integrals $I_L^{(I,0)}, I_R^{(I)}$ and dimensionless angular velocity of the bodies $d = d(\tau)$

The integral action for libration in the vicinity of the center (32) has the form of

$$I_L^{(I,1)} = \frac{2}{\sqrt{(1-x_1)(1+x_2)}}\left[\frac{(1-x_1)(1-x_2)}{(x_3-1)}K(m) + \Pi(n_1,m) + \frac{(x_1-x_3)(x_3-x_2)}{(x_3^2-1)}\Pi(n_2,m)\right]$$

where

$$m = \frac{(x_1+1)(x_2-1)}{(x_1-1)(x_2+1)}, \qquad n_1 = \frac{x_1+1}{x_1-1},$$
$$n_2 = \frac{(x_1+1)(x_3-1)}{2(x_1-1)(x_3+1)}.$$

The rotation of the intermediate gyrostat correspond to the complex-conjugate roots (78), and, in this case, the action integral is written as

$$I_R^{(I)} = \frac{2\operatorname{sign}(x_3-\nu)\,d\pi}{(b-a)\sqrt{x_3^2-1}} - 2\operatorname{sign}(d)\mu\left[\frac{(x_r^2+x_m^2)\nu^2-2x_r\nu+1}{(x_3\nu-1)\nu}K(m) + \sqrt{\frac{1}{1-m}}(\nu^2-1)\left(\frac{1}{\nu}\Pi(n_1,m_3) - \frac{(x_r-x_3)^2+x_m^2}{(x_3-\nu)(x_3\nu-1)}\Pi(n_2,m_3)\right)\right] \tag{80}$$

where

$$\mu = \frac{(\cos\theta_1\cos\theta_2)^{-1/2}}{x_m}, \qquad m = \sin^2\frac{\theta_1-\theta_2}{2},$$
$$m_3 = \frac{m}{m-1}, \qquad n_1 = \nu^2, \qquad n_2 = \left(\frac{\nu x_3-1}{\nu-x_3}\right)^2.$$

It should be noted that for the intermediate gyrostat, the imaginary numbers of the first term and the last term in (80) are of opposite sign and equal in value.

Let us consider an example of the disturbed motion of the intermediate gyrostat, where the relative dimensionless angular velocity of the bodies $d = d(\tau)$ varies as shown in Fig. 7b. The phase portrait $s = s(l)$ is shown in Fig. 7a. Figure 7b shows that the averaged action integrals $I_L^{(I,0)}$ and $I_R^{(I)}$ for the libration and rotation are adiabatic invariants, the first integrals of the disturbed system.

7 Conclusion

This paper presents the study of undisturbed and disturbed motion of the axial gyrostats.

It has been shown that the equations of motion for the axial gyrostats can be reduced to two first-order ordinary differential equations for Andoyer–Deprit canonical variables which describe disturbed motion of the axial gyrostats under the influence of a small internal angular moment. For the undisturbed motion, when the internal moment is equal to zero, the stationary solutions have been found, and their stability has been studied. We have also obtained exact general analytical solutions in terms of elliptic functions and separatrix trajectories. The bifurcation diagram and phase portraits have been constructed for three types of gyrostats: oblate, prolate and intermediate. These results can be interpreted as the development of the classical Euler case for a solid, when to one degree of freedom—the relative rotation of bodies—is added. For the disturbed motion of the gyrostats, when there is a system with slowly varying parameters, we have received the adiabatic invariants in terms of complete elliptic integrals, which are approximately equal to the first integrals of the disturbed system. The adiabatic invariants remain approximately constant along a trajectory for long time intervals during which the parameter changes considerably. By means of computer numerical simulations of a perturbed motion, we used a numerical method to check the preservation constancy of the adiabatic invariants. The results of the study can be useful for the analysis of dual-spin spacecraft dynamics and for studying the chaotic behavior of spacecraft.

Acknowledgement This research was supported by the Russian Foundation for Basic Research (09-01-00384).

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