Aerospace Science and Technology 13 (2009) 224–231

Contents lists available at ScienceDirect

Aerospace Science and Technology

www.elsevier.com/locate/aescte

Analysis of the resonance and ways of its elimination at the descent of spacecrafts in the rarefied atmosphere

Vladimir Aslanov *, Alexander Ledkov

Theoretical Mechanics Department, Samara State Aerospace University, Moskovskoye Shosse 34, 443086 Samara, Russia

ARTICLE INFO

Article history:
Received 4 February 2008
Accepted 22 April 2009
Available online 14 May 2009

Keywords:
Spacecrafts
Resonance
Averaged equations
Stability criteria
Perturbed motion

ABSTRACT

In this paper uncontrolled motions of blunt-shaped spacecrafts of small elongation descended in the rarefied atmosphere is considered. Such spacecrafts can have three balancing position of a spatial angle of attack. It can result in a resonance at change of a dynamic pressure at the descend. It is shown, that numerical integration of equations of perturbed motion does not allow to receive authentic results because of stochastic character of transients. Procedure of calculation of the upper and lower estimates of parameters of motion with use of received averaged equations and a stability criteria is developed. Efficiency of this procedure is shown on an example of hypothetical spacecraft descent. The analysis of ways of the resonance elimination is executed. In result, analytical formulas for the initial angular velocity of the spacecraft and for spacecraft's geometrical parameters are obtained.

1. Introduction

For effective braking at a descent in the rarefied atmosphere the blunt-shaped spacecrafts of small elongation are used [2,3,5,6,9]. Such spacecrafts can have three balancing positions of the angle of attack: stable position in the points $\alpha_* = 0$ and $\alpha_* = \pi$; and unstable in the third intermediate point $\alpha_* \in (0;\pi)$ [2]. Presence of three balancing positions can be a cause of appearance of three regions separated by a separatrix on a phase portrait $\dot{\alpha} = \dot{\alpha}(\alpha)$. Under an action of external disturbances the phase trajectory can transfer from one region to another. The transferring is accompanied by significant increase or diminution of the spatial angle of attack α. This phenomenon can be considered as a resonance. Falling of the phase trajectory into one of the regions depends on the current phase of the spatial angle of attack and has a stochastic nature. Therefore the direct numerical integration of equations of motion does not allow to receive full representation about behaviour of the spacecraft and a series of calculations is required. Besides, the numerical integration of the equations of motion is complicated by high-frequency character of oscillations of the angle of attack.

The problem is stated to receive the approximate average system of differential equations which are not containing fast variables; to receive a stability criterion of regions of the phase portrait; to construct the calculated procedure which allows to receive all possible alternatives of the perturbed motion.

Presence of the resonance can become a serious obstacle for successful realization of the spacecraft mission. The problem is stated to study various ways of elimination of the resonance.

2. Equations of motion of the spacecraft

We investigate the perturbed motion of the segmental-conical spacecraft at the descent in the rarefied atmosphere. As disturbances we consider the small damping torques and dynamic pressure variation at the descent. The perturbed motion of the spacecraft is described by the system of equations [3]:

$$\ddot{\alpha} + F(\alpha, z) = -\varepsilon m_3(z)\dot{\alpha}$$
$$\dot{z} = \varepsilon \Phi_z(\alpha, z) \qquad (1)$$

where

$$z = \{R, G, V, \theta, H\} \qquad q = 0.5 \cdot \rho V^2$$
$$\varepsilon\Phi_R(\alpha, z) = \varepsilon m_1(z) R$$
$$\varepsilon\Phi_G(\alpha, z) = \varepsilon\big(m_2(z)G + \big[m_1(z) - m_2(z)\big] \cdot R\cos\alpha\big)$$
$$\varepsilon\Phi_V(\alpha, z) = -c_{x\alpha}(\alpha, V) \cdot \frac{qS}{m} - g\sin\theta$$
$$\varepsilon\Phi_\theta(\alpha, z) = -\frac{\cos\theta}{V}\left(g - \frac{V^2}{R_P + H}\right)$$
$$\varepsilon\Phi_H(\alpha, z) = V\sin\theta$$
$$F(\alpha, z) = \frac{(G - R\cos\alpha)(R - G\cos\alpha)}{\sin^3\alpha} - M_\alpha(\alpha, z)$$
$$m_i(z) = m_i^{\omega_i}(z) \cdot I_i^{-1} \quad (i = 1, \ldots, 3)$$

* Corresponding author.
E-mail address: aslanov_vs@mail.ru (V. Aslanov).

1270-9638/$ – see front matter
doi:10.1016/j.ast.2009.04.007

Nomenclature

a	amplitude of the first harmonic in harmonic expansion of restoring moment coefficient
A_i	region on the phase portrait ($i = 1, 2, 3$)
b	amplitude of the second harmonic in harmonic expansion of restoring moment coefficient
$c_{x\alpha}$	drag force coefficient
c_τ	coefficient of tangential force
c_n	coefficient of normal force
D	diameter of the spacecraft
E	total energy
$\bar{E}$	average value of the total energy
f	Eq. (6)
G	projections of the angular momentum vector onto velocity direction
H	flight altitude
h_1	radius of the smallest cross-section of a conical part of the spacecraft
h_2	radius of the largest cross-section of a conical part of the spacecraft
$I = I_1 = I_2$	transverse moment of inertia
I_3	pitching moment of inertia
L	characteristic size of the spacecraft
L_c	length of a frontal spherical part of the spacecraft
L_k	relative length of a conical part of the spacecraft
m	mass of the spacecraft
m_i	projection of a small dumping moment onto the axis number i of the frame $Ox_1x_2x_3$
$m_i^{\omega j}$	rotary derivative of projection of the small dumping moment
M_α	restoring moment
q	dynamic pressure
R_P	planet's radius
R	projections of the angular momentum vector onto the longitudinal axis
S	middle cross section area
T	period of oscillations of the angle of attack
u	$= \cos(\alpha)$
u_i	root of a polynomial $f(u)$
V	spacecraft motion velocity
W	potential energy of a non-perturbed system
x_{cp}	coordinates of the center of pressure of the spacecraft concerning its nose
x_{cm}	coordinates of the center of mass of the spacecraft concerning its nose
z	vector of slowly varying parameters
α	spatial angle of attack
Λ	criterion of stability of the perturbed motion in the separatrix neighborhood
ε	small parameter
θ	trajectory inclination angle
ρ	density of the atmosphere
φ	roll orientation angle
ω	angular velocity of the spacecraft
$Ox_1x_2x_3$	right-hand reference frame, with O located at the center of mass of the spacecraft. Ox_1 is direction along the axis of a symmetry of a spacecraft, Ox_2 is perpendicular Ox_1 and lays in a plane formed by Ox_1 and a local vertical

Indices

$*$	value of function at the balancing position
m	amplitude value
max	maximum value
min	minimum value

All other parameters are described in Nomenclature.

The system (1) has been obtained with a series of assumptions: the planet was considered as not rotating full-sphere, wind perturbations and influence of a aerodynamic lifting force on the angle θ were not taken into account.

The restoring moment can be approximated as:

$$M_\alpha(\alpha, z) = A(z)\sin\alpha + B(z)\sin 2\alpha \tag{2}$$

where $A = qSLI^{-1}a$, $B = qSLI^{-1}b$. This approximation allows to take into account three balancing positions on an angle of attack.

At $\varepsilon = 0$ the perturbed system (1) is reduced to a non-perturbed system with one degree of freedom:

$$\ddot{\alpha} + F(\alpha, z) = 0 \tag{3}$$

Eq. (3) has the first integral:

$$\frac{\dot{\alpha}^2}{2} + W(\alpha) = E \tag{4}$$

where

$$W(\alpha) = \int F(\alpha)\,d\alpha = \frac{G^2 + R^2 - 2GR\cos\alpha}{2\sin^2\alpha} + A\cos\alpha + B\cos^2\alpha$$

The integral of the energy (4) after replacement $u = \cos(\alpha)$ takes the form:

$$\dot{u}^2 = f(u) \tag{5}$$

where $f(u)$ – polynomial of the fourth degree:

$$f(u) = 2bu^4 + 2au^3 - 2(b+E)u^2 + 2(RG - a)u + \left(2E - G^2 - R^2\right) \tag{6}$$

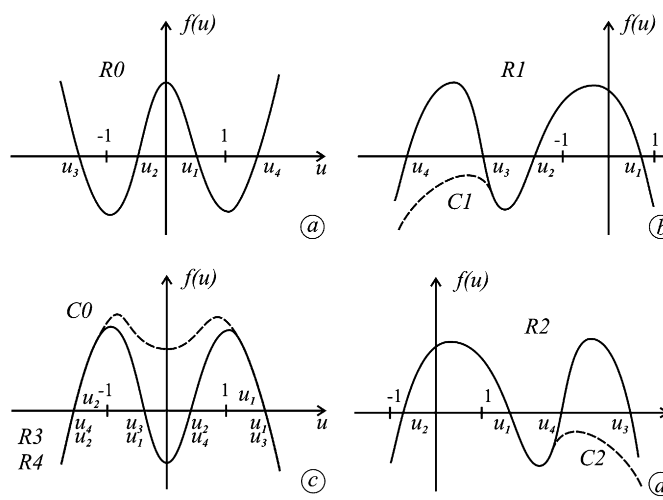

Fig. 1. Versions of a disposition of the roots of the polynomial $f(u)$.

The polynomial (6) has four roots u_1, u_2, u_3, u_4. The number of characteristic variants of a disposition of these roots is limited. Values $u \in [-1; 1]$ and non-negative values of function $f(u)$ (according to Eq. (5)) are correspond to actual mechanical process.

We consider two cases: first, when $B > 0$, and second, when $B < 0$. In the first case all roots are real, and two of them lie on the segment $[-1; 1]$. We enumerate them as follows: $u_1 > u_2$, $u_3 > u_4$, $u_1, u_2 \in [-1; 1]$ (Fig. 1a). In the second case various versions of a disposition of roots are possible. Inside the segment $[-1; 1]$ always there are two real roots. We designate them so that $u_1 > u_2$. If u_3 and u_4 are the real roots they can lie more to the left (Fig. 1b), inside (Fig. 1c), or more to the right (Fig. 1d) of the segment $[-1; 1]$. We enumerate them depending on a disposition as follows: $u_4 < u_3 < -1, -1 < u_4 < u_3 < u_2 < u_1 < 1, -1 < u_2 < u_1 < u_4 < u_3 < 1, 1 < u_4 < u_3$. If u_3 and u_4 are the complex-conjugate roots: $u_3 = u_{34} + i\nu, u_4 = u_{34} - i\nu$, three versions of their disposition are possible: $-1 < u_2 < u_{34} < u_1 < 1$, $u_{34} < -1$, $u_{34} > 1$ (Figs. 1b–1d) [3].

3. Stability criterion of perturbed motion

The phase plane of the considered system can be divided into three regions by the separatrix (Figs. 2, 3). During descent in the atmosphere the phase trajectory can move away from the separatrix, being "immersed" in current region. Also the phase trajectory can approach the separatrix, being "pushed out" from current region. The region with the first type of motion we will term stable, and with second type – unstable. Unstability of the region means, that the phase trajectory will intersect the separatrix in some finite time. At any time on the phase portrait one of two possible versions of allocation of stability is implemented [3]: 1) two regions are unstable and one is stable (Fig. 2); 2) two regions are stable and one is unstable (Fig. 3). If at intersection of the separatrix there are two stable regions on the phase portrait the further behaviour of the trajectory depends on the current phase of the angle of attack. If the phase is not determined, to fall into any region is of a random character. For choice of region in which motion will continue it is possible to use the concept of probability of "capture" [7]. This probability is determined on the basis of calculating the areas of regions encompassed by a separatrix. Analytical finding of these areas is reduced to calculation of improper integrals.

In order to estimate the stability of the regions it is not necessarily to calculate their areas. Under the action of small perturbations the average value of the total energy $\bar{E}$ slowly changes, as well as the value of potential energy W_*, calculated at the saddle point $u = u_*$. For determining the stability it is sufficient to make use of time derivatives of mentioned functions [4]. The inner region (A_1 or A_2) is stable, if the following condition is satisfied near the separatrix:

$$\dot{\bar{E}}(z) < \dot{W}(u_*, z) \tag{7}$$

For the outer region A_0 the stability condition looks like:

$$\dot{\bar{E}}(z) > \dot{W}(u_*, z) \tag{8}$$

The value of function $f(u)$ at the saddle point $u = u_*$ is equal to:

$$f_* = 2\left(1 - u_*^2\right)\left[\bar{E}(z) - W(u_*, z)\right]$$

In the neighborhood of the separatrix

$$\bar{E}(z) - W(u_*, z) = O(\varepsilon)$$

$$\dot{u}_*(z) = O(\varepsilon)$$

We calculate a derivative of last expression with respect to time and truncate terms of the order ε^2:

$$\dot{f}_* = 2\left(1 - u_*^2\right)\left[\dot{\bar{E}}(z) - \dot{W}(u_*, z)\right] + O\left(\varepsilon^2\right)$$
$$\approx 2\left(1 - u_*^2\right)\left[\dot{\bar{E}}(z) - \dot{W}(u_*, z)\right]$$

Results from this that

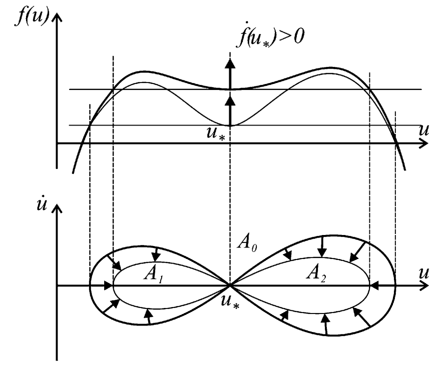

Fig. 2. Evolution of the phase portrait (A_0-stable; A_1, A_2-unstable).

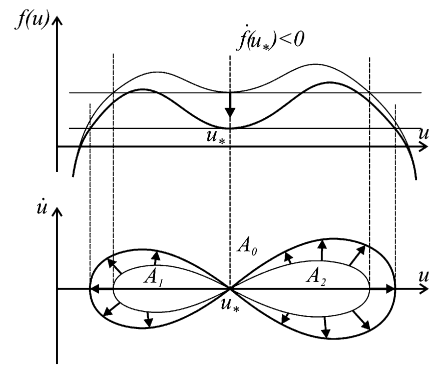

Fig. 3. Evolution of the phase portrait (A_0-unstable; A_1, A_2-stable).

$$\dot{f}_* < 0$$

$$\dot{f}_* > 0$$

correspond to conditions (7) and (8) respectively.

Indeed, if in the inner region (A_1 or A_2) the value of polynomial $f(u)$ at point u_* decreases, then this region is stable. In the opposite case the region is unstable, and the phase trajectory will not fall into it at any initial conditions. Similarly, the outer region A_0 will be stable or unstable with increasing or decreasing f_*, respectively.

The total energy of the system is equal to the potential energy calculated for the amplitude value of the angle of attack $\alpha = \alpha_{\mathrm{m}}$ (for $\dot{\alpha} = 0$). It is obvious that

$$\bar{E}(z) = W(\alpha_{\mathrm{m}}, z)$$

where α_{m} and z correspond to the averaged equations. We suppose that the averaged equations of motion corresponding to system (1) are obtained. We can calculate the derivatives $\dot{\bar{E}}(z)$ and $\dot{W}(\alpha_*, z)$ in virtue of the averaged equations:

$$\dot{\bar{E}}(z) = \left.\frac{\partial W}{\partial \alpha}\right|_{\alpha=\alpha_{\mathrm{m}}} \cdot \dot{\alpha}_{\mathrm{m}} + \sum_z \left.\frac{\partial W}{\partial z}\right|_{\alpha=\alpha_{\mathrm{m}}} \cdot \dot{z}$$
$$= F(\alpha_{\mathrm{m}}, z) \cdot \dot{\alpha}_{\mathrm{m}} + \sum_z \left.\frac{\partial W}{\partial z}\right|_{\alpha=\alpha_{\mathrm{m}}} \cdot \dot{z}$$

and

$$\dot{W}(\alpha_*, z) = \sum_z \left.\frac{\partial W}{\partial z}\right|_{\alpha=\alpha_*} \cdot \dot{z}$$

We introduce the criterion which define a stability of perturbed motion in the separatrix neighborhood [2]

$$\Lambda \equiv F(\alpha_{\mathrm{m}}, z) \cdot \dot{\alpha}_{\mathrm{m}} + \sum_z \left.\frac{\partial W}{\partial z}\right|_{\alpha_*}^{\alpha_{\mathrm{m}}} \cdot \dot{z} \tag{9}$$

or for the system (1)

$$\begin{aligned}\Lambda = {} & \frac{(G - R\cos\alpha_m)(R - G\cos\alpha_m)}{\sin^3\alpha_m} - M_\alpha(\alpha_m, z) \\ & + \left[\frac{R - G\cos(\alpha_m)}{\sin^2(\alpha_m)}\cdot m_1(z)\cdot R \right. \\ & + \frac{G - R\cos(\alpha_m)}{\sin^2(\alpha_m)}\cdot m_2(z)\cdot G \\ & \left. + \frac{dA}{dq}\frac{\dot q}{\varepsilon}\cos(\alpha_m) + \frac{dB}{dq}\frac{\dot q}{\varepsilon}\cos^2(\alpha_m)\right] \end{aligned} \tag{10}$$

and then, finally, we can write the stability conditions: for the inner regions A_1 and A_2

$$\Lambda < 0 \tag{11}$$

and for outer region A_0

$$\Lambda > 0 \tag{12}$$

Eq. (10) contains the magnitudes obtained from an average equations system. We will write this system.

4. Averaged equations

If during the descent occurs the resonance the single integration of the system (1) allows to receive only one of several possible trajectories. The region of motion in which the trajectory will pass at the resonance, depends on the current phase of the angle of attack. This phase is determined by initial conditions, therefore it is necessary to execute a series of calculations with different initial conditions for deriving all possible versions of the spacecraft motion.

At an integration of the average equations it is possible to select region in which motion after approach of the resonance will proceed. It considerably simplifies the analysis of behaviour of the spacecraft. Besides, the integration of the system (1) is hampered by presence of the fast varying variable α. The average system is deprived this shortcoming. We will receive the average system.

The solution of Eq. (3) is periodic. It reaches one maxima $\alpha = \alpha_{max}$ and one minimum $\alpha = \alpha_{min}$ on this period T. For deriving the average equations we use V.M. Volosov's method [8]. Period can be defined from the expression:

$$T = \int_0^T dt = \oint \frac{d\alpha}{\dot\alpha} = \int_{\alpha_{min}}^{\alpha_{max}} \frac{d\alpha}{\dot\alpha} - \int_{\alpha_{max}}^{\alpha_{min}} \frac{d\alpha}{\dot\alpha} = 2\int_{\alpha_{min}}^{\alpha_{max}} \frac{d\alpha}{\dot\alpha}$$

Amplitudes of oscillation α_{max} and α_{min} are related by the equality:

$$E = W(\alpha_{max}) = W(\alpha_{min})$$

From (4) follows, that

$$\dot\alpha^2 = 2\big(E - W(\alpha)\big)$$

We write a total time derivative of the integral (4):

$$\dot E = \dot\alpha\ddot\alpha + \frac{\partial W}{\partial\alpha}\dot\alpha + \sum_z \frac{\partial W}{\partial z}\dot z \tag{13}$$

Making use of the system (1) we can average the obtained equation on period T:

$$\dot E = \frac{2\varepsilon}{T}\int_{\alpha_{min}}^{\alpha_{max}} \left[m_3(z)\dot\alpha^2 + \sum_z \frac{\partial W}{\partial z}\dot z\right]\frac{d\alpha}{\dot\alpha} \tag{14}$$

We write the similar equation for amplitude of oscillation α_m. The total derivative of the function $W(\alpha)$ in the point $\alpha = \alpha_m$ is equal

$$\frac{dW}{dt} = \frac{\partial W(\alpha_m)}{\partial\alpha_m}\cdot\dot\alpha_m + \sum_z \frac{\partial W(\alpha_m)}{\partial z}\dot z$$

Taking into account, that $E = W(\alpha_m)$ and using Eq. (13), we can receive:

$$\begin{aligned}\dot\alpha_m = {} & \frac{2\varepsilon}{TF(\alpha_m)}\int_{\alpha_{min}}^{\alpha_{max}}\left[m_3(z)\dot\alpha^2 \right. \\ & \left. + \sum_z\left(\frac{\partial W}{\partial z} - \frac{\partial W(\alpha_m)}{\partial z}\right)\dot z\right]\frac{d\alpha}{\dot\alpha}\end{aligned} \tag{15}$$

Let us expand brackets in expressions (14) and (15) in view of expression for $W(\alpha)$:

$$\begin{aligned}\dot E = {} & \frac{2\varepsilon}{T}\Big[m_3(z)J_1 + \big(R^2 m_2(z) + G^2 m_2(z)\big)J_2 - 2RGm_2(z)J_3 \\ & + R^2\big(m_1(z) - m_2(z)\big)\frac{T}{2} + \frac{dA}{dq}\frac{\dot q}{\varepsilon}J_4 + \frac{dB}{dq}\frac{\dot q}{\varepsilon}J_5\Big]\end{aligned} \tag{16}$$

$$\begin{aligned}\dot\alpha_m = {} & \frac{2\varepsilon}{TF(\alpha_m)}\Big[m_3(z)J_1 + \big(R^2 m_2(z) + G^2 m_2(z)\big)J_2 \\ & - 2RGm_2(z)J_3 + R^2\big(m_1(z) - m_2(z)\big)\frac{T}{2} + \frac{dA}{dq}\frac{\dot q}{\varepsilon}J_4 \\ & + \frac{dB}{dq}\frac{\dot q}{\varepsilon}J_5\Big] - \frac{2\varepsilon}{TF(\alpha_m)}\Big[\frac{R - G\cos(\alpha_m)}{\sin^2(\alpha_m)}\cdot m_1(z)\cdot R \\ & + \frac{G - R\cos(\alpha_m)}{\sin^2(\alpha_m)}\cdot m_2(z)\cdot G + \frac{dA}{dq}\frac{\dot q}{\varepsilon}\cos(\alpha_m) \\ & + \frac{dB}{dq}\frac{\dot q}{\varepsilon}\cos^2(\alpha_m)\Big]\frac{K(k)}{\beta}\end{aligned} \tag{17}$$

where

$$J_1 = \int_{\alpha_{min}}^{\alpha_{max}} \dot\alpha\, d\alpha \qquad J_2 = \int_{\alpha_{min}}^{\alpha_{max}} \frac{d\alpha}{\dot\alpha\sin^2\alpha} \qquad J_3 = \int_{\alpha_{min}}^{\alpha_{max}} \frac{\cos\alpha\, d\alpha}{\dot\alpha\sin^2\alpha}$$

$$J_4 = \int_{\alpha_{min}}^{\alpha_{max}} \frac{\cos\alpha\, d\alpha}{\dot\alpha} \qquad J_5 = \int_{\alpha_{min}}^{\alpha_{max}} \frac{\cos^2\alpha\, d\alpha}{\dot\alpha}$$

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k^2\sin^2 t}}\,dt$$

The integrals J_i $(i = 1, \ldots, 5)$ can be reduced to complete normal elliptic Legendre integrals of the first, second, and third kind. For this purpose we should make use of one of changes of variables [3]:

$$u = \frac{u_1(u_2 - u_3) + u_3(u_1 - u_2)\cos^2\gamma}{(u_2 - u_3) + (u_1 - u_2)\cos^2\gamma} \tag{18}$$

$$u = \frac{(u_2 + u_1\xi) - (u_2 - u_1\xi)\cos\gamma}{(1 + \xi) - (1 - \xi)\cos\gamma} \tag{19}$$

where $\xi = \frac{\cos\chi_1}{\cos\chi_2}$, $\operatorname{tg}\chi_1 = \frac{u_1 - u_{34}}{\nu}$, $\operatorname{tg}\chi_2 = \frac{u_2 - u_{34}}{\nu}$; also we should take into account that

$$d\alpha = -\frac{du}{\sqrt{1 - u^2}} \qquad \dot\alpha = \sqrt{\frac{f(u)}{1 - u^2}} \qquad \frac{du}{\sqrt{f(u)}} = -\frac{d\gamma}{\beta\sqrt{1 - k^2\sin^2\gamma}}$$

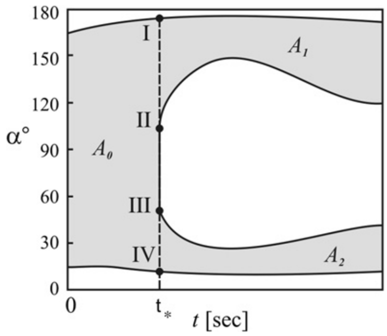

Fig. 4. Enveloping oscillations of the spatial angle of attack.

Expression (18) and written below equalities needs to be used, when all roots of the polynomial $f(u)$ are real (R0, R1, R2, R3, R4)

$$k=\sqrt{\frac{(u_1-u_2)\cdot(u_3-u_4)}{(u_1-u_3)\cdot(u_2-u_4)}}\qquad \beta=\sqrt{\frac{1}{2}b(u_1-u_3)\cdot(u_4-u_2)}$$

Expression (9) and written below equalities needs to be used, when a complex roots of the polynomial $f(u)$ are presence (C0, C1, C2)

$$k=\frac{1}{2}\left(1-\frac{(u_1-u_{34})(u_2-u_{34})+\nu^2}{\eta}\right)\qquad \beta=\sqrt{2B\eta}$$

$$\eta^2=\left[(u_1-u_{34})^2+\nu^2\right]\cdot\left[(u_2-u_{34})^2+\nu^2\right]$$

We find the averaged expressions for slowly varying variables:

$$\dot{R}=\varepsilon m_1(z)R$$

$$\langle\dot{G}\rangle=\frac{2\varepsilon}{T}\int\limits_{\alpha_{\min}}^{\alpha_{\max}}\left(m_2(\tau)G+\left[m_1(\tau)-m_2(\tau)\right]R\cos\alpha\right)\frac{d\alpha}{\dot{\alpha}}$$

$$=\varepsilon m_2(\tau)G+\frac{2\varepsilon}{T}\left[m_1(\tau)-m_2(\tau)\right]RJ_5$$

$$\langle\dot{V}\rangle=\frac{2}{T}\int\limits_{\alpha_{\min}}^{\alpha_{\max}}\left(-c_{x\alpha}(\alpha,V)\cdot\frac{qS}{m}-g\sin\theta\right)\frac{d\alpha}{\dot{\alpha}}$$

$$=-\frac{2qS}{mT}\int\limits_{\alpha_{\min}}^{\alpha_{\max}}c_{x\alpha}(\alpha,V)\frac{d\alpha}{\dot{\alpha}}-g\sin\theta$$

$$=-\frac{2qS}{mT}\int\limits_{\alpha_{\min}}^{\alpha_{\max}}\frac{c_{x\alpha}(\alpha,V)\,d\alpha}{\sqrt{2(E-W(\alpha))}}-g$$

$$\sin\theta=\frac{2qS}{mT}\int\limits_{u_1}^{u_2}\frac{c_{x\alpha}(\arccos u,V)\,du}{\sqrt{2(E-W(\arccos u))(1-u^2)}}-g\sin\theta$$

$$\dot{\theta}=-\frac{\cos\theta}{V}\left(g-\frac{V^2}{R_P+H}\right)$$

$$\dot{H}=V\sin\theta \tag{20}$$

where $\langle\cdots\rangle$ denote averaging.

Thus, Eqs. (17) and (20) form the closed average system of the differential equations of the first order. Use of the average equations (17), (20) allows to increase considerably speed of calculation in comparison with the system (1).

At numerical integration of the average system it is possible to choose current region. For this purpose it is necessary to define the order of numbering of roots of the polynomial $f(u)$ and the variable taken as a α_{m}. In Fig. 4 are shown enveloping oscillations of the spatial angle of attack. In the instant t_* the phase trajectory intersects the separatrix. Thus motion can proceed both in region A_1, and in region A_2. To calculate motion in A_1, it is necessary to use equalities $u_1=\cos\alpha_{\mathrm{II}}, u_2=\cos\alpha_{\mathrm{I}}, \alpha_{\mathrm{m}}=\alpha_{\max}=\alpha_{\mathrm{I}}$; and to calculate motion in $A_2-u_1=\cos\alpha_{\mathrm{IV}}, u_2=\cos\alpha_{\mathrm{III}}, \alpha_{\mathrm{m}}=\alpha_{\min}=\alpha_{\mathrm{IV}}$.

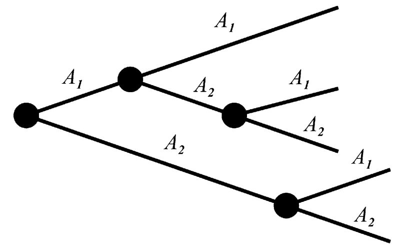

Fig. 5. The scheme of the some motion.

5. Calculated procedure

On the basis of the carried out analysis of perturbed motion of the spacecraft with biharmonic coefficient of the restoring moment it is possible to offer calculation procedure of the upper and lower estimations of motion parameters with use of the averaged equations.

If on the phase portrait there are three regions, in a neighborhood of the separatrix (A_0, A_1 and A_2) three variants of a disposition of the $f(u)$ polynomial roots are realized accordingly (C0, R4 and R3, Fig. 1). The numerical integration of the averaged equations (17), (20) is being fulfilled from an initial point belonging to one of regions till the moment of intersection of the separatrix. At the moment of intersection of the separatrix one of transitional variants is realized: R3–C0 or C0–R4. Then for each of region (A_0, A_1 and A_2) under the formula (10) the criterion Λ is being calculated, and according to conditions (11) and (12) their stability is being estimated.

We can mark some the obvious facts. If the outer region A_0 is stable, regions A_1 and A_2 – are unstable and vice versa. Transition from A_1 in A_2 and from A_2 in A_1 is possible only through A_0. The region from which there was an exit on the separatrix, always is unstable, therefore always there are either one, or two stable regions. In the first case the stable region for prolongation of the integration is being selected. In the second case the task has probability character and for deriving the upper and lower estimations of the solution the calculations are being carried out for each of stable regions.

Schematically motion can be presented as a tree which titles of branches correspond to variants of motion chosen at intersection through the separatrix. Very simply to organize sequential bypass of such tree (Fig. 5).

6. Elimination of the resonance by the choice of the form and the position of the mass center of the spacecraft

The resonance arises in the time of intersection of the phase trajectory of the separatrix, therefore for its elimination it is necessary to provide absence of the separatrix and the saddle point on the phase portrait of the system. In [2] it is shown, that a necessary condition of absence of the saddle point is the condition

$$|b|\leqslant 0.5|a|$$

It is obvious, that if this inequality is fulfilled, function $m_\alpha(\alpha)$ possess the value of zero on the segment $[0,\pi]$ only in two points: $\alpha=0$ and $\alpha=\pi$ (Fig. 6).

Aerodynamic characteristics of an axisymmetric rigid body, as a rule, set by the instrumentality of three associations on the angle of attack: a coefficient of tangential force c_τ, a coefficient of normal force c_n and a position of the center of pressure x_{cp} [1].

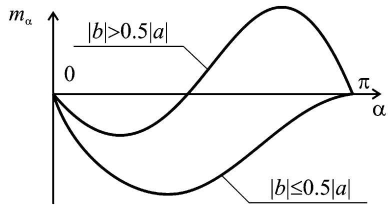

Fig. 6. Dependence of the coefficient restoring moment on the angle of attack.

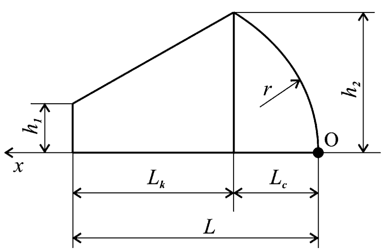

Fig. 7. Configuration of the spacecraft.

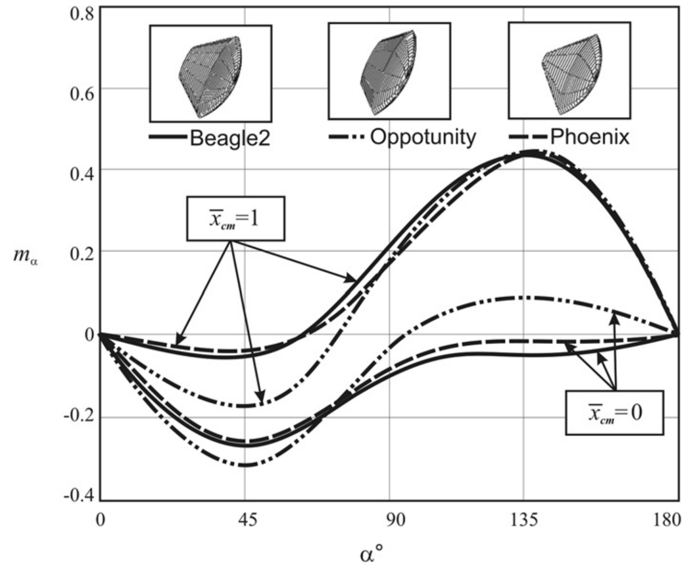

Fig. 8. Boundaries of area of a disposition of curves $m_\alpha = m_\alpha(\alpha)$ for the spacecrafts such as Beagle2, Opportunity and Phoenix.

Frequently instead of coordinate of the center of pressure the coefficient of the moment concerning a nose of the rigid body m_O is used. Then the restoring moment coefficient concerning mass center of the rigid body can be defined by the formula

$$m_\alpha = -c_n(\bar{x}_{cp} - \bar{x}_{cm}) = m_O + c_n\bar{x}_{cm}$$

where $\bar{x}_{cp} = x_{cp}/L$, $\bar{x}_{cm} = x_{cm}/L$. If the graph of dependence of the coefficient of the restoring moment concerning the nose of the spacecraft lays below an abscissa axis

$$m_O(\alpha) < 0$$

for coefficient of the restoring moment concerning center of mass it is possible to provide absence of the third balancing position by the instrumentality of interior arrangement. Values $\bar{x}_{cm} = 0; 1$ define boundaries of area for all $m_\alpha(\alpha, \bar{x}_{cm})$ curves. If the lower boundary of the indicated area $m_O = m_\alpha(\alpha, 0)$ lays below the abscissa axis, that, obviously, it is possible to find such value $\bar{x}_{cm}$ that the curve $m_\alpha(\alpha)$ laid below the abscissa axis. Hence, due to interior arrangement it is possible to exclude the intermediate balancing position for the spacecraft of the given shape.

We will consider a rigid body, consisting of a frontal spherical cap, a conical segment and a flat bottom (Fig. 7). Spacecrafts of the similar configuration were used in the European and American martian programs: Mars Exploration Rover (Spirit, Opportunity), Mars Express (Beagle2) [9], Phoenix Mars Mission (Fig. 8). As varied geometrical parameters of the spacecraft we will use $\bar{h}_1$ – the relative radius of the least cross-section of the conical part, $\bar{L}_c$ – the relative length of the frontal spherical part, $\bar{L}_k$ – the relative length of the conical part. All parameters are referred to diameter of the spacecraft $D = 2h_2$. We impose the geometrical parameters limitations

$$0 \leqslant \bar{h}_1 < 0.5 \qquad \bar{L}_k > 0 \qquad 0 < \bar{L}_c < 0.5$$

For definition of aerodynamic coefficients we will take advantage of the percussion theory of Newton [1]. For hypersonic speeds at motion in the rarefied atmosphere this theory yields outcomes well compounded with experimental data. In Fig. 8 are shown obtained by the method of Newton of association for spacecrafts such as Beagle2, Opportunity and Phoenix for boundary values of centerings $\bar{x}_{cm} = 0; 1$.

If derivatives $m_O(\alpha)$ on α in points $\alpha = 0$ and $\alpha = \pi$ have a different sign

$$\left.\frac{dm_O(\alpha)}{d\alpha}\right|_{\alpha=0} \cdot \left.\frac{dm_O(\alpha)}{d\alpha}\right|_{\alpha=\pi} < 0 \tag{21}$$

the lower boundary of area for the spacecraft of the considered configuration has only two balancing positions ($\alpha = 0$ and $\alpha = \pi$).

At calculation of aerodynamic coefficients by the Newton's method in a neighborhood of the point $\alpha = 0$ the frontal part of the spacecraft influences on them. Other parts of the spacecraft do not influence coefficients as they are in the field of an aerodynamic shadow. In a neighborhood of the point $\alpha = \pi$ the conical part influences only. The flat part does not create normal aerodynamic force, and the spherical part is in the field of the aerodynamic shadow.

Making use of the Newton's method, we will write the following expressions for $m_O(\alpha)$ in a neighborhood of points $\alpha = 0, \pi$ [1]

$$m_O(\alpha)|_{\alpha\approx 0} = -\frac{k(M)}{D^2}\frac{2h_2^4}{Lr}\cos\alpha\sin\alpha \tag{22}$$

$$m_O(\alpha)|_{\alpha\approx\pi} = \frac{k(M)\sin(2\alpha)(h_2 - h_1)L_k}{D^2 2L((h_2 - h_1)^2 + L_k^2)} \times \left(\frac{(h_2 + 2h_1)L_k^2}{3} + \frac{2(h_1^3 - h_2^3)}{3} + L_k L_c(h_2 + h_1)\right) \tag{23}$$

where $k(M)$ – coefficient of pressure of inhibiting action behind a normal shock wave [1], M – Mach number. We calculate a derivative of function (22) on the angle of attack in the point $\alpha = 0$

$$\left.\frac{dm_O(\alpha)}{d\alpha}\right|_{\alpha=0} = -\frac{k(M)}{D^2}\frac{2h_2^4}{Lr} < 0$$

This derivative always the negative (i.e. the balancing position always stable), therefore a requirement (21) can be rewritten as

$$\left.\frac{dm_O(\alpha)}{d\alpha}\right|_{\alpha=\pi} > 0 \tag{24}$$

Taking into account expressions (23) and (24) the condition of absence of the third balancing position (21) after some transformations may be written as

$$f_O(L_c, L_k, h_1) = (h_2 + 2h_1)L_k^2 + 2\big(h_1^3 - h_2^3\big) + 3L_k L_c(h_2 + h_1) > 0$$

The surface $f_O(L_c, L_k, h_1) = 0$ divides space of three variables (L_c, L_k, h_2) into two parts (Fig. 9). All points located below to this

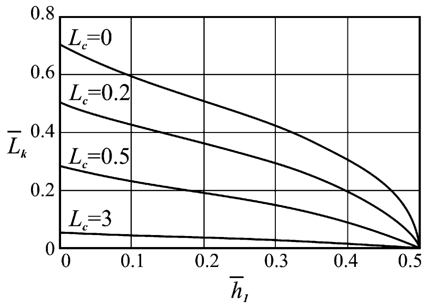

Fig. 9. Cuts of the surface $f(L_c, L_k, h_2) = 0$ by the planes, parallel to plane.

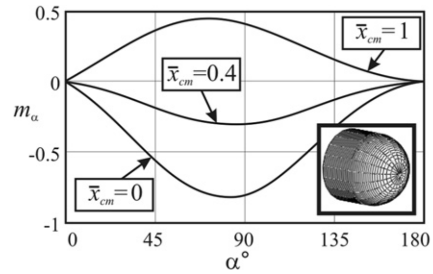

Fig. 10. Spacecraft with a large gamut of the possible position of the mass center.

surface correspond to the spacecrafts having three balancing positions, irrespective of their interior arrangement. Opportunity Lander is example of such spacecraft (Fig. 8). For all points located above of this surface, absence of the third balancing position is possible to achieve by the instrumentality of interior arrangement.

We will prove, that for the spacecraft of considered configuration (Fig. 7) the case when all area m_α lays below the abscissa axis is impossible. We write expressions similar (22) and (23) for the upper bound of area. For this purpose we use the formula (21) and we accept $\bar{x}_{cm} = 1$.

$$m_\alpha(\alpha)\big|_{\substack{\alpha\approx 0\\ \bar{x}_{cm}=1}} = \frac{k(M)}{D^2}\frac{2h_2^4(L-r)}{Lr^2}\cos\alpha\sin\alpha \tag{25}$$

$$m_\alpha(\alpha)\big|_{\substack{\alpha\approx \pi\\ \bar{x}_{cm}=1}} = -\frac{k(M)\sin(2\alpha)(h_2-h_1)L_k}{D^2 6L((h_2-h_1)^2+L_k^2)} \times \left(2\left(h_2^3-h_1^3\right)+(2h_2+h_1)L_k^2\right) \tag{26}$$

The upper bound of area lays below the abscissa axis if the following condition is fulfilled

$$\frac{dm_\alpha(\alpha)}{d\alpha}\bigg|_{\substack{\alpha=0\\ \bar{x}_{cm}=1}} = \frac{k(M)}{D^2}\frac{2h_2^4(L-r)}{Lr^2} < 0$$

i.e. $L < r$. Then the condition of existence of two balancing positions can be written as

$$\frac{dm_\alpha(\alpha)}{d\alpha}\bigg|_{\substack{\alpha=\pi\\ \bar{x}_{cm}=1}} > 0$$

Calculating a derivative (26) in the point $\alpha = \pi$ and fulfilling some transformations, we receive

$$f_1(L_c, L_k, h_1) = -2\left(h_2^3 - h_1^3\right) - (2h_2 + h_1)L_k^2 > 0 \tag{27}$$

Since for considered configuration $h_2 > h_1$, $f_1(L_c, L_k, h_1) < 0$ at any values of parameters L_c, L_k, h_2. Hence the requirement (27) is never fulfilled and the upper bound of area always has a part above of the abscissa axis.

On the foundation of the fulfilled analysis it is possible to give recommendations at the choice of parameters of the shape of the spacecraft for elimination of the resonance. From the back side of the spacecraft it is necessary to use the long cone with largest radius h_1, and from its frontal side – the spherical surface of the smallest radius r (Fig. 10).

7. Elimination of the resonance by selection of initial conditions of motion

In article [2] connection between coefficients in expansion of moment (2) and existence of stable and unstable positions of the equilibrium on the phase portrait of the non-perturbed system (3) is established. It is shown, that if the condition

$$b \geqslant -\frac{I}{2SLq}\min_{-1\leqslant u\leqslant 1} W_g''(u) \equiv b_* \tag{28}$$

where

$$W_g''(u) = \frac{(G^2+R^2)(1+3u^2) - 2GRu(3+u^2)}{(1-u^2)^3} \tag{29}$$

is fulfilled, the potential energy of a system $W(\alpha, z)$ on the interval $[0; \pi]$ has one local minimum. It means, that on the phase portrait of the system there is a unique stable position of the equilibrium, and no saddle points. According to (28) and (29), it is possible to provide absence of the saddle point by the selection enough large, but finite values of R and G. In practice it is possible to achieve by a fast twirling of the spacecraft around of its long axis.

Substituting in Eq. (28) the minimum of function (29) we receive

$$b_* = \begin{cases} -\frac{I}{8SLq}\frac{R^4(G^2-R^2)}{[(G^2-R^2)-\sqrt{G(G^2-R^2)}]^2} & \text{when } G > R \\ -\frac{I}{8SLq}\frac{-G^4(G^2-R^2)}{[(G^2-R^2)+\sqrt{-R(G^2-R^2)}]^2} & \text{when } G < R \end{cases} \tag{30}$$

Let us consider, that in an initial instant $\omega_y = \omega_z = 0$. Magnitudes R and G are related to angular velocities of the spacecraft by equalities:

$$R = \frac{I_x\omega_x}{I}$$

$$G = \frac{I_x\omega_x}{I}\cos\alpha + (\omega_z\sin\varphi - \omega_y\cos\varphi)\sin\alpha$$

Substituting these expressions in (30), we receive

$$b_* = -\frac{I_x^2\omega_{x0}^2 p(\alpha)}{8SLqI} \tag{31}$$

where

$$p(\alpha) = \begin{cases} (\cos(2\alpha) + \operatorname{sign}(\omega_x)\sin(2\alpha))^{-1} & \text{when } G > R \\ 1 + \sin^2\alpha + 2\sin\alpha & \text{when } G < R \end{cases}$$

In the case when $R = G$, potential energy $W(u)$ has a unique local minimum and there is only one region of motion on the phase portrait.

Eq. (28) can be written as

$$b \geqslant -\frac{I_x^2\omega_{x0}^2 p(\alpha)}{8SLqI}$$

Whence it follows that if the initial angular velocity satisfies the inequality

$$\omega_{x0} \geqslant \sqrt{-b\frac{8SLqI}{I_x^2 p(\alpha)}}$$

there is no saddle point on the phase portrait.

If during motion parameters R and G vary insignificantly, the critical value (31) decreases by modulo due to magnification of the dynamic pressure. During the unguided descent in the atmosphere the dynamic pressure has a unique well-defined maxima $q_{\max}$ [3].

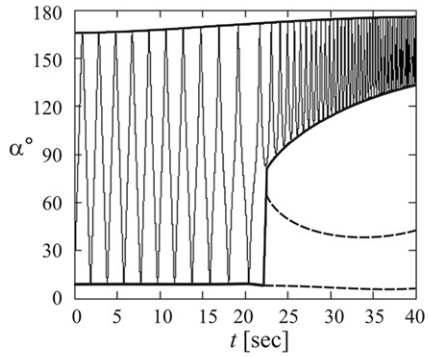

Fig. 11. Dependence of the angle of attack on a time and envelopes of the angle of attack.

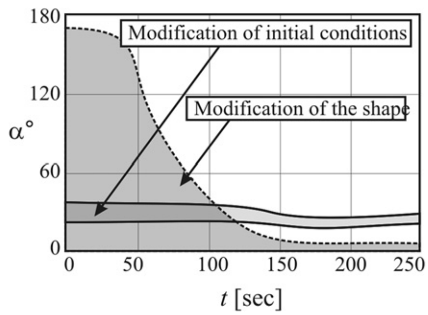

Fig. 12. Envelopes of the angle of attack.

Therefore as an initial value of the angular velocity it is possible to take

$$\omega_{x0} > \sqrt{-b\frac{8SLq_{\max}I}{I_x^2 p(\alpha_0)}} \tag{32}$$

It guarantees absence of a saddle point on a phase portrait during all descent. The rough value $q_{\max}$ can be received after calculation of motion of the center of mass of the spacecraft at the descent in the atmosphere without taking into account motion concerning center of mass.

8. Numerical simulation

As an example we consider motion of a hypothetical spacecraft, having the following geometrical parameters: $\bar{r} = 0.75, \bar{h}_2 = 0.5, \bar{h}_1 = 0.2, \bar{L}_k = 0.381, \bar{L}_c = 0.191, \bar{x}_{cm} = 0.7$. Its mass equals 70 kg, and coefficients $a = 0.11$, $b = -0.192$. The descent happens in the Martian atmosphere. Initial conditions are equals $\alpha_0 = 30°$, $\dot{\alpha}_0 = 0$, $R_0 = 0.2\ \text{m}^{-1}$, $G_0 = 0.7\ \text{m}^{-1}$, $V_0 = 5000$ m/sec, $\theta_0 = -15°$, $H_0 = 1.2 \times 10^5$ m. Fig. 11 shows dependence of the angle of attack on a time, obtained as the result of the numerical integration of the system (28). In this figure also envelopes of the angle of attack obtained by the instrumentality of the calculated procedure given in Section 5 are shown. The resonance occurs at $t = 23$ sec. The phase trajectory intersects the separatrix and passes from unstable outer region of motion in one of interior. In the case shown in figure the phase trajectory has passed in the upper region. Envelop of the lower region where the phase trajectory also could hit, is shown in figure by a dotted line.

We consider two methods of elimination of the resonance: by modification of the shape of the spacecraft and by the selection of the initial conditions of motion. In the first case we increase radius h_1 of the conical part of the spacecraft and we shift center of masses closer to the nose of the spacecraft $\bar{x} = 0.4$ (Fig. 10). During descent in the atmosphere the oscillation amplitude of the angle of attack decreases, and their frequency is incremented. Oscillations happen around of the unique stable balancing position $\alpha = 0$. In the second case as the initial angular velocity ω_{x0} we choose the magnitude, satisfying the inequality (32). In the second case the oscillation frequency of the angle of attack is more than in first case (Fig. 12). During the descent the amplitude varies insignificantly.

9. Conclusions

Thus, we have investigated the perturbed motion of the spacecraft with biharmonic moment characteristic at the descent in the rarefied atmosphere. It has shown, that exist transitive modes (resonance) at which parameters of motion considerable change. Criteria of stability of motion in the separatrix neighborhood is obtained. Procedure of calculation of the upper and lower estimates of parameters of motion with use of received averaged equations and a stability criteria is offered.

We have investigated means of elimination of the transitive modes. Influence of the shape and position of center of mass of a spacecraft on a resonance is analysed. By the instrumentality of the percussion theory of Newton the analytical criterion of absence of the resonance is obtained. The possibility of elimination of a resonance by selection of initial conditions of motion is shown. The analytical formula providing absence of a resonance is found for the initial angular velocity of the spacecraft. The problem was considered in ideal statement without taking into account a variation of aerodynamic characteristics from Mach number, altitude and other parameters, however it is possible to assume, that the established tendencies are maintained and in more common statement.

Acknowledgements

The authors would like to thank the Russian Fund for Basic Research (Project Nos. 06-01-00355a, 09-01-00384-a) which supported this work.

References

[1] N.S. Arjannikov, G.S. Sadekova, Aerodinamika Letatelnih Apparatov (Aerodynamics of a Flying Vehicle), Higher School, Moscow, 1983.
[2] V.S. Aslanov, A.S. Ledkov, Features of rotational motion of a spacecraft descending in the martian atmosphere, Cosmic Research (ISSN 0010-9525) 45 (4) (2007) 331–338.
[3] V.S. Aslanov, Prostranstvennoe Dvizhenie Tela pri Spuske v Atmosfere (Spatial Motion of a Body at Descent in the Atmosphere), Fizmatlit, Moscow, 2004, 160 pp.
[4] V.S. Aslanov, Resonance at motion of a body in the Mars's atmosphere under biharmonical moment, WSEAS Transaction on Systems and Control 3 (1) (January 2008) 33–39.
[5] J. Collinet, P. Brenner, S. Palerm, Dynamic stability of the HUYGENS probe at Mach 2.5, Aerospace Science and Technology 11 (2–3) (2007) 202–210.
[6] G. Colombatti, A. Aboudan, F. Ferri, F. Angrilli, Huygens probe entry dynamic model and accelerometer data analysis, Planetary and Space Science 56 (5) (2008) 601–612.
[7] A.I. Neishtadt, Issues of the perturbation theory for nonlinear resonance systems, Doctoral (Phys.-Math.) Dissertation, VINITI, Moscow, 1988.
[8] V.M. Volosov, Some types of calculations using averaging in the nonlinear vibration theory, Zh. Vychisl. Mat. Mat. Fiz. 3 (1) (1963) 3–53.
[9] A. Wilson, A. Chicarro, ESA SP-1240: Mars Express: the Scientific Payload, ESA Publications Division, ISBN 92-9092-556-6, 2004.