Dynamics and control of dual-spin gyrostat spacecraft with changing structure

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Abstract

We study the motion of the free dual-spin gyrostat spacecraft that consists of the platform with a triaxial ellipsoid of inertia and the rotor with a small asymmetry with respect to the axis of rotation. The system with perturbations caused by a small asymmetry of the rotor and the time-varying moments of inertia of the rotor is considered. The dimensionless equations of the system are written in Serret-Andoyer canonical variables. The system's phase space is described. It is shown that changes in the moments of inertia of the gyrostat leads to the deformation of the phase space. The internal torque control law is proposed that keeps the system at the center point in the phase space. The effectiveness of the control is shown through a numerical simulation. It's shown that the uncontrolled gyrostat can lose its axis orientation. Proposed internal torque keeps the initial angle between the axis of the gyrostat and the total angular momentum vector.

Keywords: Variable structure gyrostat spacecraft, Serret-Andoyer variables, Internal torque, Stabilization control

Nomenclature

$\tilde{\mathbf{g}}_\delta$	Dimensional internal torque
$\boldsymbol{a}$	Unit vector of the gyrostat axis
$O_r x_r y_r z_r$	Rotor's principal frame

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$O_p x_p y_p z_p$	Platform's principal frame
$Ox'_p y'_p z'_p$	Frame fixed at the gyrostat center of mass with the axes parallel to the platform's principal axes
$Ox'_r y'_r z'_r$	Frame fixed at the gyrostat center of mass with the axes parallel to the rotor's principal axes
$\mathbf{G}$	Total angular momentum vector of the gyrostat
$\boldsymbol{\omega}$	Angular velocity of the platform
G_x, G_y, G_z	Projections of the total angular momentum vector onto the axes Ox'_p, Oy'_p, Oz'_p respectively
$\omega_x, \omega_y, \omega_z$	Projections of the platform angular velocity vector onto the axes Ox'_p, Oy'_p, Oz'_p respectively
$\mathbf{J}(t)$	Inertia tensor of the gyrostat
$\tilde{G}_r$	Angular momentum of the rotor relative to the rotation axis
$\mathbf{J}_p$	Tensor of inertia of the platform
$\mathbf{J}_r(t)$	Tensor of inertia of the rotor
$\tilde{A}_p, \tilde{B}_p, \tilde{C}_p$	Moments of inertia of the platform in the frame $O_p x_p y_p z_p$
$\tilde{A}_r, \tilde{B}_r, \tilde{C}_r$	Moments of inertia of the rotor in the frame $O_r x_r y_r z_r$
A_p, B_p, C_p	Moments of inertia of the platform in the frame $Ox'_p y'_p z'_p$
A_r, B_r, C_r	Moments of inertia of the rotor in the frame $Ox'_p y'_p z'_p$
l, L, G	Subset of Serret-Andoyer canonical variables
τ	Dimensionless time

1. Introduction

Artificial satellites can contain one or more spinning rotors to provide gyroscopic stability of a desired orientation of the vehicle. Dual-spin satellites use the spin of a rotor to maintain pointing accuracy of an antenna platform or a solar sail. Some types of satellites use small but rapidly spinning momentum wheels to control the attitude of a large platform. Gyrostats and gyrostat satellite motions are investigated in a large number of papers (Wittenburg, 1977; Hall and Rand, 1994; Hall, 1995; Cochran et al., 1982; Elipe and Lanchares, 2008; Aslanov, 2012, 2011; Sarychev and Mirer, 2001; Lanchares et al., 1998; Shirazi and Ghaffari-Saadat, 2005; Aslanov and Doroshin, 2010), Wittenburg in (Wittenburg, 1977) introduced the equations of a multibody gyrostat and presented analytical solutions for the free gyrostat . Hall and Rand (1994); Hall (1995) investigated the dynamical behavior of an axially symmetric gyrostat under the action of slight internal angular torque.

Cochran et al. (1982) found analytical solution for a asymmetric gyrostat in dynamical variables and solutions for Euler angles in quadratures. Elipe and Lanchares (2008) found exact analytical solutions for the problem of the attitude dynamics of a free gyrostat. Aslanov (2012) built dimensionless equations of the axial gyrostat and got analytical solutions in terms of Jacobi elliptic functions for different ellipsoids of inertia and different modes of the motion. Sarychev and Mirer (2001) investigated the stability of the satellite-gyrostat stationary motion. Lanchares et al. (1998) considered the dual-spin deformable spacecraft with time-dependent moments of inertia as periodic function of time. Stabilization is obtained by means of a spinning rotor.

Our paper focused on the study of the dual-spin spacecraft with time-dependent moments of inertia (Lanchares et al., 1998) as a *general* function of time. The moments of inertia of the gyrostat change slowly with time which may be related to deployment of solar panels, solar sails or any other constructions. In this case the satellite gyrostat can change its type from prolate to oblate or vice versa (Aslanov, 2012). The internal control torque can be used to provide the stable motion of the gyrostat. The purpose of this paper is *to find the control law* that keeps the stable motion in the vicinity of the equilibrium position.

The paper divided into five sections. In the section 1 the statement of the problem is given. In the section 2 the motion of the axial gyrostat as two rigid bodies connected by a rigid shaft is considered. The dynamics of the gyrostat is described by ordinary differential equations in therms of the Serret-Andoyer canonical variables (Deprit, 1967). The dimensionless equations are provided based on (Aslanov, 2012). In the section 3 the phase space of the gyrostat is described. It's shown that the time varying moments of inertia of the rotor lead to the changes in the phase space topology. The section 4 proposes the internal control torque that maintains the position of the system in phase space at the center point, therefore the angle between the gyrostat axis and the angular momentum vector remains constant. Effectiveness of the control is shown in section 5 by means of a numerical simulation.

2. Equations of motion

2.1. Euler equations

Let us consider a spatial motion of a gyrostat spacecraft around the center of mass in the absence of external torque. The gyrostat spacecraft consists of two bodies: the rotor and the platform, connected by a cylindrical joint. The axis of the cylindrical joint, the principal axis z_p of the platform and the principal axis z_r of the rotor are parallel to the unit vector $\boldsymbol{a}$. The orientation of the frame $O_r x_r y_r z_r$ relative to the platform's frame $O_p x_p y_p z_p$ is defined by the angle δ, fig. 1. The internal torque $\tilde{\mathbf{g}}_\delta$ acts on the platform and on the rotor:

$$\tilde{\mathbf{g}}_\delta = \boldsymbol{a}\tilde{g}_\delta. \tag{1}$$

where $\tilde{g}_\delta$ is the internal torque value.

Derivation of the equations of motion is based upon the theorem of changing the total angular momentum vector of the gyrostat

$$\frac{d\mathbf{G}}{dt} = \frac{\tilde{d}\mathbf{G}}{dt} + \boldsymbol{\omega} \times \mathbf{G} = 0, \tag{2}$$

where $\boldsymbol{\omega}$ is the platform's angular velocity, $\mathbf{G}$ is the total angular momentum vector

$$\mathbf{G} = \mathbf{J}(t) \cdot \boldsymbol{\omega} + \tilde{G}_r \boldsymbol{a}, \tag{3}$$

$\tilde{d}\mathbf{G}/dt$ is the local derivative of the $\mathbf{G}$ in the frame $O_p x_p y_p z_p$, $\mathbf{J}(t)$ is the time dependent tensor of inertia of the gyrostat composed of the platform's tensor of inertia $\mathbf{J}_p$ and the rotor's tensor of inertia $\mathbf{J}_r(t)$

$$\mathbf{J}(t) = \mathbf{J}_p + \mathbf{J}_r(t), \tag{4}$$

$\tilde{G}_r$ is the angular momentum of the rotor relative to the platform.

$$\tilde{G}_r = \tilde{C}_r \dot{\delta}, \tag{5}$$

$\tilde{C}_r$ is the moment of inertia of the rotor relative to the axis $\boldsymbol{a}$.

Next we project equation (2) onto the axes Ox'_p, Oy'_p, Oz'_p. The axes x'_p, y'_p, z'_p are parallel to the corresponding axes of the frame $O_p x_p, y_p, z_p$. The platform's inertia tensor in the frame $O_p x_p y_p z_p$ is denoted by $\tilde{\mathbf{J}}_p$. The rotor's inertia tensor in the frame $O_r x_r y_r z_r$ is denoted by $\tilde{\mathbf{J}}_r$. We suppose that $\tilde{\mathbf{J}}_p$ and $\tilde{\mathbf{J}}_r$ both are diagonal matrices and $\tilde{\mathbf{J}}_r$ depends on time

$$\tilde{\mathbf{J}}_p = diag(\tilde{A}_p, \tilde{B}_p, \tilde{C}_p),\ \tilde{\mathbf{J}}_r(t) = diag(\tilde{A}_r(t), \tilde{B}_r(t), \tilde{C}_r(t)). \tag{6}$$

Further, for shortness we omit (t) from any parameters that depend on $\tilde{\mathbf{J}}_r$. Let us write the rotor's inertia tensors and the platform's inertia tensors in the frame $Ox'_p y'_p z'_p$ (Wittenburg, 1977)

$$\mathbf{J}_p = \tilde{\mathbf{J}}_p + \left(\mathbf{r}_{pc}^T \mathbf{r}_{pc} \mathbf{E} - \mathbf{r}_{pc} \mathbf{r}_{pc}^T\right) m_p, \tag{7}$$

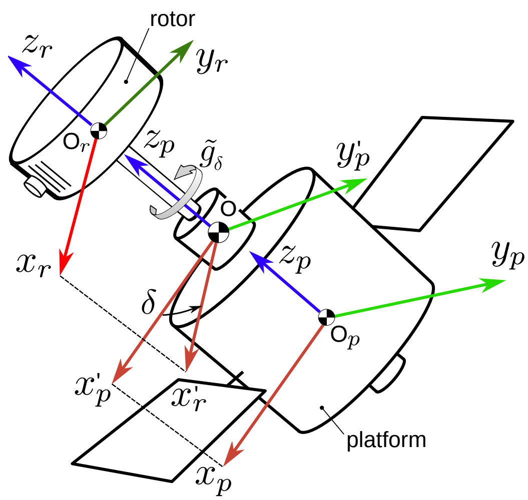


Figure 1: The coordinate frames

$$\mathbf{J}_r = \mathbf{A}_{pr}\left(\widetilde{\mathbf{J}}_r + \left(\mathbf{r}_{\mathrm{rc}}^T\mathbf{r}_{\mathrm{rc}}\mathbf{E} - \mathbf{r}_{\mathrm{rc}}\mathbf{r}_{\mathrm{rc}}^T\right)m_r\right)\mathbf{A}_{pr}^T, \tag{8}$$

where m_p, m_r are the masses of the platform and the rotor respectively. The vectors $\overrightarrow{O_pO} = \mathbf{r}_{pc}$ and $\overrightarrow{O_rO} = \mathbf{r}_{rc}$ (fig. 1) are:

$$\mathbf{r}_{pc} = \begin{pmatrix}0 & 0 & z_{pc}\end{pmatrix}^T, \ \mathbf{r}_{rc} = \begin{pmatrix}0 & 0 & z_{rc}\end{pmatrix}^T. \tag{9}$$

The well-known orthogonal matrix $\mathbf{A}_{pr}$ transforms coordinates from the frame $Ox'_ry'_rz'_r$ to the frame $Ox'_py'_pz'_p$

$$\mathbf{A}_{pr} = \begin{pmatrix}\cos\delta & -\sin\delta & 0\\ \sin\delta & \cos\delta & 0\\ 0 & 0 & 1\end{pmatrix}. \tag{10}$$

$\mathbf{A}_{pr}^T$ is the transposed matrix $\mathbf{A}_{pr}$. Taking into account (7), we obtain the platform's inertia tensor in the frame $Ox'_py'_pz'_p$

$$\mathbf{J}_p = \begin{pmatrix} m_pz_{pc}^2 + \tilde{A}_p & 0 & 0\\ 0 & m_pz_{pc}^2 + \tilde{B}_p & 0\\ 0 & 0 & \tilde{C}_p\end{pmatrix}. \tag{11}$$

Plug (10), (6) into (8), we get the rotor's inertia tensor in the frame $Ox'_p y'_p z'_p$

$$\mathbf{J}_r = \begin{pmatrix} \tilde{A}_r \cos^2\delta + m_r z_{rc}^2 + \tilde{B}_r \sin^2\delta & (\tilde{A}_r - \tilde{B}_r)\cos\delta\sin\delta & 0 \\ (\tilde{A}_r - \tilde{B}_r)\cos\delta\sin\delta & \tilde{B}_r \cos^2\delta + m_r z_{rc}^2 + \tilde{A}_r \sin^2\delta & 0 \\ 0 & 0 & \tilde{C}_r \end{pmatrix}. \tag{12}$$

The moments of inertia of the rotor A_r, B_r, C_r in the frame Ox'_r, y'_r, z'_r are defined as

$$\tilde{\mathbf{J}}_{rc} = \begin{pmatrix} m_r z_{rc}^2 + \tilde{A}_r & 0 & 0 \\ 0 & m_r z_{rc}^2 + \tilde{B}_r & 0 \\ 0 & 0 & \tilde{C}_r \end{pmatrix} = \begin{pmatrix} A_r & 0 & 0 \\ 0 & B_r & 0 \\ 0 & 0 & C_r \end{pmatrix}.$$

The inertia tensor of the rotor in the frame $Ox'_p y'_p z'_p$ is

$$\mathbf{J}_r = \mathbf{A}_{pr}\tilde{\mathbf{J}}_{rc}\mathbf{A}_{pr}^T = \begin{pmatrix} A_r\cos^2\delta + B_r\sin^2\delta & (A_r - B_r)\sin\delta\cos\delta & 0 \\ (A_r - B_r)\sin\delta\cos\delta & A_r\sin^2\delta + B_r\cos^2\delta & 0 \\ 0 & 0 & C_r \end{pmatrix}.$$

Now we can construct the total inertia tensor of the gyrostat

$$\mathbf{J} = \begin{pmatrix} A_p + A_r \cos^2\delta + B_r \sin^2\delta & (A_r - B_r)\sin\delta\cos\delta & 0 \\ (A_r - B_r)\sin\delta\cos\delta & B_p + A_r \sin^2\delta + B_r \cos^2\delta & 0 \\ 0 & 0 & C_p + C_r \end{pmatrix} = \begin{pmatrix} J_x & -J_{xy} & 0 \\ -J_{xy} & J_y & 0 \\ 0 & 0 & J_z \end{pmatrix}. \tag{13}$$

The total angular momentum vector $\mathbf{G}$ in the frame $O'_p x'_p y'_p z'_p$ has the following coordinates

$$\mathbf{G} = (G_x, G_y, G_z). \tag{14}$$

Substituting (14), (13) in (2), we get

$$\begin{cases} \dot{G}_x = \dfrac{G_y\left(J_{xy}^2 - J_x J_y\right)\tilde{G}_r - G_z\left(G_x J_z J_{xy} + G_y\left(J_{xy}^2 + J_x\left(J_z - J_y\right)\right)\right)}{J_z\left(J_x J_y - J_{xy}^2\right)} \\ \dot{G}_y = \dfrac{G_z\left(G_y J_z J_{xy} + G_x\left(J_{xy}^2 + J_y\left(J_z - J_x\right)\right)\right)}{J_z(J_x J_y - J_{xy}^2)} + \dfrac{G_x\tilde{G}_r}{J_z} \\ \dot{G}_z = \dfrac{G_x G_y\left(J_x - J_y\right) + \left(G_x^2 - G_y^2\right)J_{xy}}{J_x J_y - J_{xy}^2} \end{cases} \tag{15}$$

Note that the equations (15) depend on the angle δ. These equations should be supplemented by the equation for the angle δ.

For the axisymmetric rotor the inertia tensor of the gyrostat (13) is diagonal $A_r = B_r$

$$\mathbf{J} = \begin{pmatrix} A_p + A_r & 0 & 0 \\ 0 & B_p + A_r & 0 \\ 0 & 0 & C_p + C_r \end{pmatrix} = \begin{pmatrix} J_x & 0 & 0 \\ 0 & J_y & 0 \\ 0 & 0 & J_z \end{pmatrix}.$$

In this case the equations (15) have the known form (Hughes, 1986; Cochran et al., 1982; Hall and Rand, 1994):

$$\begin{cases} \dot{G}_x = \dfrac{G_y G_z (J_y - J_z)}{J_y J_z} - \dfrac{G_y \tilde{G}_r}{J_z}, \\ \dot{G}_y = \dfrac{G_x G_z (J_z - J_x)}{J_x J_z} + \dfrac{G_x \tilde{G}_r}{J_z}, \\ \dot{G}_z = \dfrac{G_x G_y (J_x - J_y)}{J_x J_y}. \end{cases} \tag{16}$$

The equations (15) and (16) have the obvious integral

$$G = \sqrt{G_x^2 + G_y^2 + G_z^2} = const.$$

since they are derived from the equation (2).

2.2. Serret-Andoyer variables

Let us write the equations (15) in terms of the Serret-Andoyer canonical variables (Deprit, 1967) for the rotor with a small asymmetry

$$\varepsilon = \frac{A_r - B_r}{A_r} \ll 1. \tag{17}$$

The components of the total angular momentum vector $\mathbf{G} = const$ in terms of the canonical variables (l, L, G) are expressed as (fig. 2)

$$G_x = \sqrt{G^2 - L^2} \sin l, \tag{18}$$

$$G_y = \sqrt{G^2 - L^2} \cos l, \tag{19}$$

$$G_z = L, \tag{20}$$

Let us substitute (18)–(20) into (15), taking into account a small asymmetry (17). After expanding the right hand side of the differential equations in powers of ε and after discarding terms of higher order than the first, we obtain

$$\dot{l} = \frac{L\left((B_p - A_p)\cos 2l - (A_p + B_p + 2A_r)\right)}{2\,(A_p + A_r)\,(B_p + A_r)} + \frac{L - \dot{\delta} C_r}{C_p + C_r} \qquad + \varepsilon F_l, \tag{21}$$

$$\dot{L} = \frac{(G^2 - L^2)(A_p - B_p)\sin l \cos l}{(A_p + A_r)(A_r + B_p)} \qquad + \varepsilon F_L, \tag{22}$$

where

$$F_l = -\frac{L A_r\left[(A_p + A_r)\cos\delta\cos l - (A_r + B_p)\sin\delta\sin l\right]^2}{(A_p + A_r)^2 (A_r + B_p)^2}, \tag{23}$$

$$F_L = \frac{A_r (G^2 - L^2)}{2(A_p + A_r)^2 (A_r + B_p)^2}\left[2A_p\cos\delta(A_r\sin(\delta + 2l) + B_p\sin\delta\cos 2l) + 2A_r B_p\sin\delta\cos(\delta + 2l) + A_p^2\cos^2\delta\sin 2l + A_r^2\sin 2(\delta + l) - B_p^2\sin^2\delta\sin 2l\right]. \tag{24}$$

2.3. Platform equation

The equations (21)–(22) should be supplemented by the equation for the angle δ. Let us write the angular momentum theorem for the platform

$$\frac{\tilde{d}\mathbf{G}_p}{dt} + \boldsymbol{\omega} \times \mathbf{G}_p = -\tilde{\mathbf{g}}_\delta + \mathbf{M}_c, \tag{25}$$

where $\mathbf{M}_c$ is a reaction torque vector ($\mathbf{M}_c \cdot \boldsymbol{a} = 0$); $\mathbf{G}_p$ is the platform angular momentum vector

$$\mathbf{G}_p = \mathbf{J}_p \cdot \boldsymbol{\omega}. \tag{26}$$

$\tilde{d}\mathbf{G}_p/dt$ is the local derivative of the $\mathbf{G}_p$ in the frame $Ox'_p y'_p z'_p$. Scalar product of the equation (25) and the vector $\boldsymbol{a}$ produces

$$\dot{G}_{pz} = \left(\frac{1}{B_p} - \frac{1}{A_p}\right) G_{px} G_{py} - \tilde{g}_\delta. \tag{27}$$

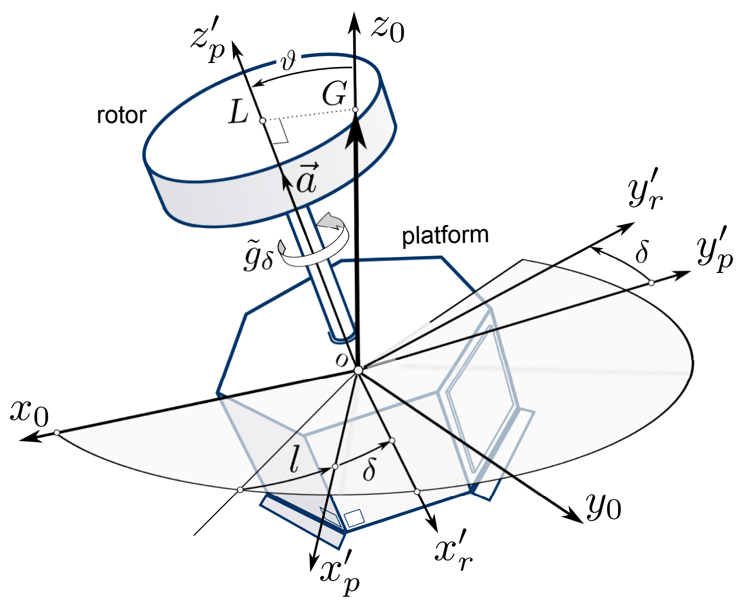


Figure 2: Serret-Andoyer variables l, L

In the terms of the platform's angular velocities components $\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)$ the equation (27) has the well known form

$$C_p\dot{\omega}_z + (B_p - A_p)\omega_x\omega_y = -\tilde{g}_\delta. \tag{28}$$

The components of the angular momentum vector of the gyrostat (3) in terms of $\omega_x, \omega_y, \omega_z$ are written as

$$\begin{cases} G_x = \omega_x(A_p + A_r\cos^2\delta + B_r\sin^2\delta) + \frac{1}{2}\omega_y(A_r - B_r)\sin 2\delta, \\ G_y = \omega_y(A_r\sin^2\delta + B_p + B_r\cos^2\delta) + \frac{1}{2}\omega_x(A_r - B_r)\sin 2\delta, \\ G_z = C_p\omega_z + C_r(\dot{\delta} + \omega_z). \end{cases} \tag{29}$$

Equating (29) with (18)–(20) we get

$$\omega_x = \frac{\sqrt{G^2 - L^2}(B_p\sin l - A_r\sin\delta\cos(\delta + l) + B_r\cos\delta\sin(\delta + l))}{D_\omega}, \tag{30}$$

$$\omega_y = \frac{\sqrt{G^2 - L^2}(A_p\cos l + A_r\cos\delta\cos(\delta + l) + B_r\sin\delta\sin(\delta + l))}{D_\omega}, \tag{31}$$

$$\omega_z = \frac{L - \dot{\delta}C_r}{C_r + C_p}. \tag{32}$$

where

$$D_\omega = B_r(A_r + B_p \sin^2\delta) + A_r B_p \cos^2\delta + A_p(A_r \sin^2\delta + B_p + B_r\cos^2\delta).$$

Substitute for $\omega_x, \omega_y, \omega_z$ in (28) using (30)-(32) and taking into account (22) we get

$$\ddot{\delta} = \frac{(G^2 - L^2)(B_p - A_p)\sin 2l}{2C_p(A_r + A_p)(A_r + B_p)} + \tilde{g}_\delta \frac{C_r + C_p}{C_p C_r} - \frac{\dot{C}_r(C_p\dot{\delta} + L)}{C_r(C_p + C_r)} + \varepsilon F_\delta, \quad (33)$$

where

$$F_\delta = \frac{A_r(G^2 - L^2)}{C_p C_r (A_r + A_p)^2 (A_r + B_p)^2} \left[(A_r + A_p)\cos\delta\cos l - (A_r + B_p)\sin\delta\sin l\right] \times$$
$$\left[C_r(B_p - A_p)\sin(l - \delta) + C_p((B_p + A_r)\cos\delta\sin l + \right.$$
$$\left.(A_p + A_r)\sin\delta\cos l)\right]. \quad (34)$$

Equation (33) can be written as a system of two first-order differential equations for $\dot{\delta}$ and for the generalized momentum p_δ. The generalized momentum p_δ is defined as

$$p_\delta = \frac{dT}{d\dot{\delta}} = C_r(\omega_z + \dot{\delta}) \quad (35)$$

where T is the kinetic energy of the gyrostat

$$2T = \boldsymbol{\omega} \cdot \tilde{\mathbf{J}}_p \cdot \boldsymbol{\omega} + \boldsymbol{\omega}_r \cdot \tilde{\mathbf{J}}_r \cdot \boldsymbol{\omega}_r$$

$\boldsymbol{\omega}_r$ is the absolute angular velocity of the rotor

$$\boldsymbol{\omega}_r = \mathbf{A}_{pr}^T \boldsymbol{\omega} + \boldsymbol{a}\dot{\delta}$$

Using (32) we can express $\dot{\delta}$ from the equation (35)

$$\dot{\delta} = \frac{p_\delta - L}{C_p} + \frac{p_\delta}{C_r} \quad (36)$$

Substituting (36) into the equation (33) we get differential equation for p_δ

$$\dot{p}_\delta = \varepsilon \frac{A_r\left(G^2 - L^2\right)}{2\left(A_p + A_r\right)^2\left(A_r + B_p\right)^2}\left[A_p B_p \cos 2\delta \sin 2l + \right.$$
$$+ 2A_r\left(A_p \cos l \sin(2\delta + l) + B_p \sin l \cos(2\delta + l)\right) +$$
$$\left. + A_p^2 \sin 2\delta \cos^2 l + A_r^2 \sin(2(\delta + l)) - B_p^2 \sin 2\delta \sin^2 l\right] + \tilde{g}_\delta \quad (37)$$

Hence, the motion of the gyrostat spacecraft with a small asymmetry of the rotor is fully described by the equations (21), (22), (36) and (36).

2.4. Dimensionless equations

Let us introduce the dimensionless parameters

$$a = \frac{C_p}{A_p + A_r},\ b = \frac{C_p}{B_p + A_r},\ c = \frac{C_p}{C_p + C_r},\ w = \frac{A_r}{C_p}, \tag{38}$$

the dimensionless generalized momentums, the internal torque

$$s = \frac{L}{G},\ d = \frac{C_r(L + C_p\dot{\delta})}{G(C_p + C_r)},\ g_\delta = \frac{C_p}{G^2}\tilde{g}_\delta \tag{39}$$

and the dimensionless time

$$\tau = \frac{G}{C_p}t. \tag{40}$$

Let us define the Hamiltonian as

$$H = H_0 + \varepsilon H_1, \tag{41}$$

where

$$H_0 = \frac{1 - s^2}{4}[a + b + (b - a)\cos 2l] + \frac{s^2}{2} - sd + \frac{d^2}{2(1 - c)} - W_\delta, \tag{42}$$

$$H_1 = -\frac{1}{2}w(s^2 - 1)\left[b\cos\delta\cos l - a\sin\delta\sin l\right]^2, \tag{43}$$

where $W_\delta = \int_{\delta_0}^{\delta} g_\delta d\delta$ is the work done by the internal torque. The equations (21), (22), (36) and (37) get the form

$$l' = \frac{\partial H_0}{\partial s} + \varepsilon\frac{\partial H_1}{\partial s}, \tag{44}$$

$$s' = -\frac{\partial H_0}{\partial l} - \varepsilon\frac{\partial H_1}{\partial l}, \tag{45}$$

$$\delta' = \frac{\partial H_0}{\partial d} + \varepsilon\frac{\partial H_1}{\partial d}, \tag{46}$$

$$d' = -\frac{\partial H_0}{\partial \delta} - \varepsilon\frac{\partial H_1}{\partial \delta}, \tag{47}$$

where $(\ )' = d(\ )/d\tau$. The partial derivatives of the Hamiltonian are defined as

$$\frac{\partial H_0}{\partial s} = s - d - \frac{s}{2}(a + b + (b-a)\cos 2l),$$
$$\frac{\partial H_0}{\partial l} = -\frac{1}{2}(b-a)(1-s^2)\sin 2l,$$
$$\frac{\partial H_0}{\partial d} = d/(1-c) - s,$$
$$\frac{\partial H_0}{\partial \delta} = -g_\delta,$$
$$\frac{\partial H_1}{\partial d} = 0,$$
$$\frac{\partial H_1}{\partial s} = -ws(b\cos\delta\cos l - a\sin\delta\sin l)^2,$$
$$\frac{\partial H_1}{\partial l} = -\frac{1}{4}w\left(1-s^2\right)\left[\sin 2l\left(b^2 - a^2 + (a^2+b^2)\cos 2\delta\right) + 2ab\sin 2\delta\cos 2l\right],$$
$$\frac{\partial H_1}{\partial \delta} = -\frac{1}{4}w\left(1-s^2\right)\left[\sin 2\delta\left(b^2 - a^2 + (a^2+b^2)\cos 2l\right) + 2ab\cos 2\delta\sin 2l\right].$$

The equations (44)-(47) describe the motion of the gyrostat in the dimensionless form.

3. System's phase space for the axisymmetric rotor

When $\varepsilon = 0$ and $g_\delta = 0$ the equations (44)-(47) can be written as

$$l' = s - d - s(a\sin^2 l + b\cos^2 l), \tag{48}$$
$$s' = \frac{1}{2}(b-a)(1-s^2)\sin 2l, \tag{49}$$
$$d = const. \tag{50}$$

with the Hamiltonian (Aslanov, 2012)

$$H = \frac{1-s^2}{4}[a + b + (b-a)\cos 2l] + \frac{s^2}{2} - sd = h = const. \tag{51}$$

There are several types of motions of the unperturbed system ($\varepsilon = 0$, $g_\delta = 0$). Each type is determined by the initial conditions and the moments of inertia of the gyrostat (Aslanov, 2012). The ratios of the moments of inertia determine three basic types of the gyrostat: oblate, prolate and intermediate. For the oblate gyrostat, the moment of inertia of the platform along the axis $\boldsymbol{a}$ (C_p) is greater than the moments of inertia of the gyrostat along any axis perpendicular to the axis $\boldsymbol{a}$ ($a > 1, b > 1$). For the prolate gyrostat, the platform's moment of inertia C_p is less than the gyrostat's total moments of inertia along any axis perpendicular to the axis $\boldsymbol{a}$. For the intermediate gyrostat the value of C_p lies between the minimum and maximum transverse moments of inertia of the gyrostat ($a < 1 < b$).

The phase space topology of the gyrostat is shown at fig.3. The motion type areas in the phase space are divided by separatrices with $h = h_s$. Full set of gyrostat types are described in (Aslanov, 2011). Here we give table 2 that contains description of the all gyrostat types. The first and the last phase space maps (1a, 5b) in table 2 characterized by one separatrix with saddles $s_s \leq 1$. The phase space maps 1b-3a and 3c-5a have one separatrix and saddles with $s_s = 1$. The phase space map 3b has two separatrices with saddles $s_s = \pm 1$.

4. Gyrostat stabilization

Let us consider the gyrostat with the axisymmetric rotor ($\varepsilon = 0$). The moments of inertia of the rotor are time-varying functions $A_r = A_r(\tau), C_r = C_r(\tau)$. The equations of the gyrostat are written as

$$l' = s - d - s(a \sin^2 l + b \cos^2 l), \tag{52}$$

$$s' = \frac{1}{2}(b - a)(1 - s^2) \sin 2l, \tag{53}$$

$$d' = g_\delta. \tag{54}$$

where the a, b and g_δ are the functions of the dimensionless time τ

$$a = \frac{C_p}{A_p + A_r(\tau)},\ b = \frac{C_p}{B_p + A_r(\tau)},\ c = \frac{C_p}{C_p + C_r(\tau)}$$

If the moments of inertia of the gyrostat are changed, the phase space of the gyrostat is deformed (saddles and centers change their position) and the uncontrolled gyrostat can lose it's initial orientation.

Here we set up the problem to retain the initial angle ϑ between the gyrostat axis $\boldsymbol{a}$ and the total angular momentum vector $\mathbf{G}$

$$\cos \vartheta_* = \cos \vartheta_0 = s_0 = const \tag{55}$$

Table 2: Gyrostat types

#	Gyrostat type	Dynamic conditions	Saddles	Centers
1a	Oblate $C_p > A > B$ $b > a > 1$	$\lvert d/(1-a)\rvert < 1$	$l_s = \frac{\pi}{2} \pm k\pi,\ k \in N$ $s_s = \frac{d}{1-a}$	$l_c = 0 \pm k\pi$ $s_c = \frac{d}{1-b}$
1b	Oblate $C_p > A > B$ $b > a > 1$	$\lvert d/(1-a)\rvert \geq 1$	$\cos 2l_s = \frac{2-a-b+2d}{b-a}$ $s_s = -\text{sgn}\ d$	$l_c = 0$ $s_c = \frac{d}{1-b}$
2	Oblate-Intermediate $C_p = A > B$ $b > a = 1$	$a = 1$	$\cos 2l_s = \frac{2-a-b+2d}{b-a}$ $s_s = -\text{sgn}\ d$	$l_c = 0$ $s_c = \frac{d}{1-b}$
3a	Intermediate $A > C_p > B$ $b > 1 > a$	$\lvert d/(1-a)\rvert \geq 1$	$\cos 2l_s = \frac{2-a-b+2d}{b-a}$ $s_s = -\text{sgn}\ d$	$l_c = 0$ $s_c = \frac{d}{1-b}$
3b	Intermediate $A > C_p > B$ $b > 1 > a$	$\lvert d/(1-a)\rvert < 1$ $\lvert d/(1-b)\rvert < 1$	$\cos 2l_s = \frac{2-a-b\pm 2d}{b-a}$ $s_s = \pm\text{sgn}\ d$	$l_{c1} = 0 \pm k\pi$ $s_{c1} = \frac{d}{1-b}$ $l_{c2} = \frac{\pi}{2} \pm k\pi$ $s_{c2} = \frac{d}{1-a}$
3c	Intermediate $A > C_p > B$ $b > 1 > a$	$\lvert d/(1-b)\rvert \geq 1$	$\cos 2l_s = \frac{2-a-b-2d}{b-a}$ $s_s = \text{sgn}\ d$	$l_c = \frac{\pi}{2} \pm k\pi$ $s_c = \frac{d}{1-a}$
4	Prolate-Intermediate $A > C_p = B$ $b = 1 > a$	$b = 1$	$\cos 2l_s = \frac{2-a-b-2d}{b-a}$ $s_s = \text{sgn}\ d$	$l_c = \frac{\pi}{2} \pm k\pi$ $s_c = \frac{d}{1-a}$
5a	Prolate $A > C_p = B$ $1 > b > a$	$\lvert d/(1-b)\rvert \geq 1$	$\cos 2l_s = \frac{2-a-b-2d}{b-a}$ $s_s = \text{sgn}\ d$	$l_c = \frac{\pi}{2} \pm k\pi$ $s_c = \frac{d}{1-a}$
5b	Prolate $A > B > C_p$ $1 > b > a$	$\lvert d/(1-b)\rvert < 1$	$l_s = 0 \pm k\pi$ $s_s = \frac{d}{1-b}$	$l_c = \frac{\pi}{2} \pm k\pi$ $s_c = \frac{d}{1-a}$

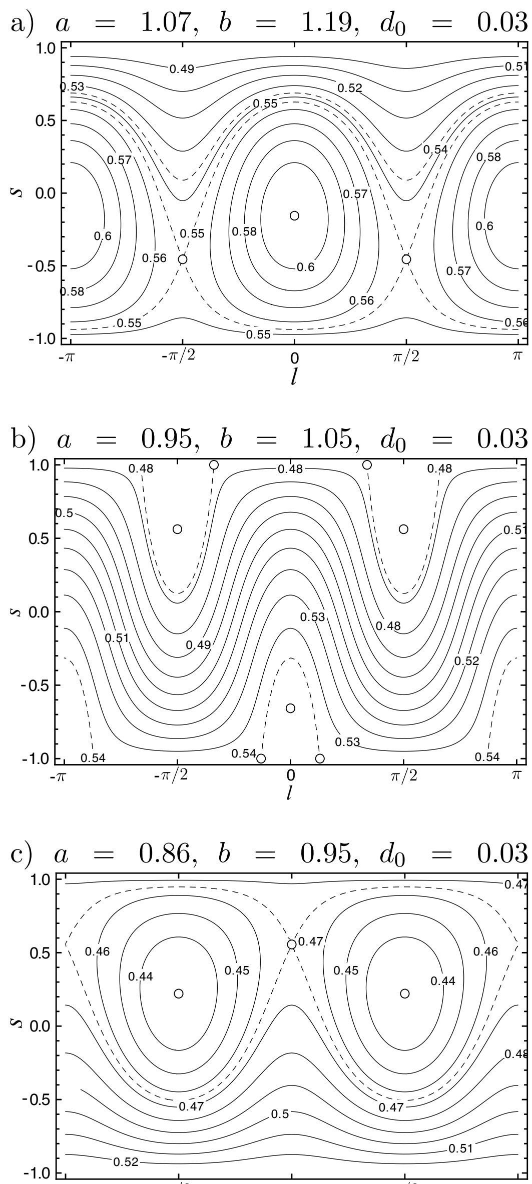


Figure 3: The energy levels for the oblate (a), intermediate (b) and the prolate (c) gyrostat

The main idea of the stabilization method is based on the conservation of the stable position by means of the internal torque g_δ. To find this torque it is suffices to differentiate the stationary solutions with respect to the dimensionless time. For the oblate gyrostat ($b > a > 1$) the stationary point in the phase space is defined as

$$s_c = s_* = \frac{d}{1-b} = \cos\vartheta_*, \tag{56}$$

then

$$d = (1-b)\cos\vartheta_* \tag{57}$$

After differentiating (57) we get

$$d' = g_\delta^{obl} = (1-b)'\cos\vartheta_*$$

or in the dimensional parameters

$$g_\delta^{obl} = \frac{C_p A_r'}{(B_p + A_r)^2}\cos\vartheta_*. \tag{58}$$

If the initial conditions of the gyrostat correspond to the stable stationary point $s_0 = s_*, l_0 = l_*$ then the control torque (58) keeps s and l constant.

By a similar way for the prolate gyrostat we get

$$s_c = s_* = \frac{d}{1-a} = \cos\vartheta_*,$$

then

$$g_\delta^{pr} = \frac{C_p A_r'}{(A_p + A_r)^2}\cos\vartheta_*. \tag{59}$$

5. Numerical examples

5.1. Gyrostat parameters

To confirm the efficiency of the proposed control let us consider several numerical examples. Suppose that the rotor has a deployable structure (e.g. solar array or solar sail). This leads to the time-dependent moments of inertia of the rotor. We assume the rotor moments of inertia A_r, C_r decrease linearly in time τ (fig. 4.a) but the gyrostat retains its type:

$$A_r(\tau) = A_{r0} - k_A\tau, \quad C_r(\tau) = k_C A_r(\tau)$$

There is a symmetry transformation for axial dual-spin spacecraft (Hall, 1992) that allows prolate gyrostat to be treated as oblate, therefore we can consider the prolate or the oblate gyrostat type only. Parameters of the gyrostat are presented in table 3. We consider the prolate gyrostat: $A > B > C_p$. The behavior of the axisymmetric prolate gyrostat is considered in the dimensionless time interval $[\tau_0; \tau_1], \tau_0 = 0, \tau_1 = 500$. For τ_0 and $\varepsilon = 0$ we have

Table 3: Parameters of the gyrostat

Parameter	Value	Parameter	Value
A_p	0.8	B_p	0.7
C_p	1.0	G	1.0
A_{r0}	1.0	k_A	0.0013
k_C	0.3		

$$A_{r0} = B_{r0} = 1.0,\ C_{r0} = 0.3,\ a_0 = 0.556,\ b_0 = 0.588,\ c_0 = 0.769,$$

and for τ_1 we have

$$A_{r1} = B_{r1} = 0.35,\ C_{r1} = 0.105,\ a_1 = 0.870,\ b_1 = 0.952,\ c_1 = 0.905,$$

Fig. 4.b shows how the dimensionless parameters a, b are changing with the dimensionless time τ.

All the cases for which a numerical simulation is performed are shown in table 4.

Table 4: Six cases for numerical simulation

Case	g_δ	ε	s_0	l_0	d_0
1	0	0	0.5	$\pi/2$	0.222
2			0.9		0.400
3	g_δ^{pr}	0	0.5	$\pi/2$	0.222
4			0.9		0.400
5	g_δ^{pr}	0.01	0.5	$\pi/2$	0.222
6			0.9		0.400

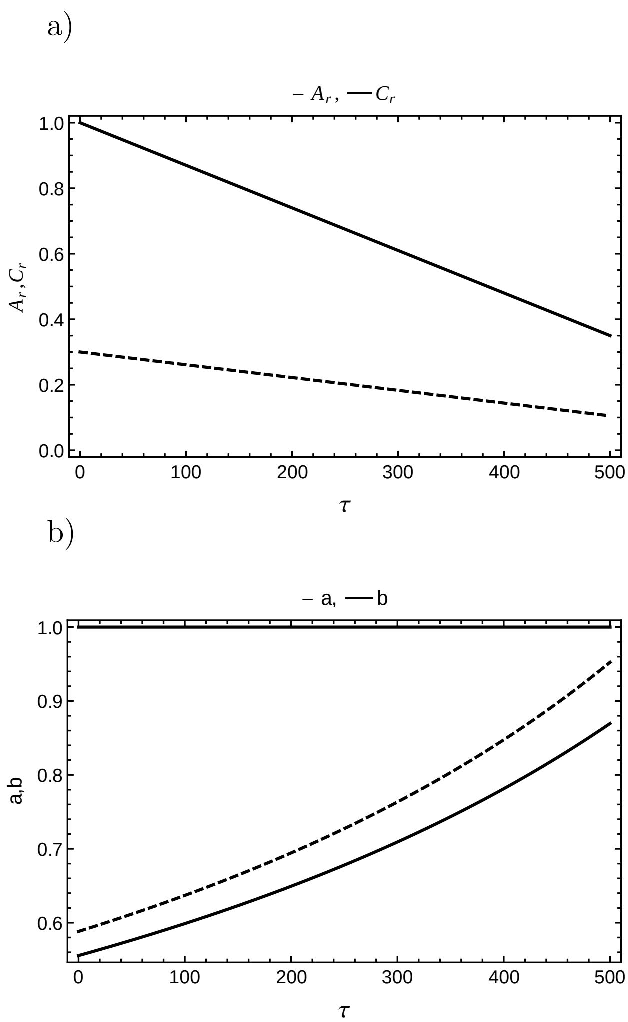


Figure 4: The moments of inertia of the rotor and the dimensionless parameters a and b

5.2. Uncontrolled motion of the prolate gyrostat with the axisymmetric rotor (the cases 1 and 2)

At first we consider the motion of the uncontrolled gyrostat with the following initial conditions correspond to the position at the center point in the phase space

$$s_0 = s_* = 0.5,\ l_0 = \frac{\pi}{2},\ d_0 = (1 - a_0)s_0,\ \delta_0 = 0. \tag{60}$$

Fig. 5.a shows how the angle $\Delta\vartheta = \vartheta - \vartheta_0$ changes with the dimensionless time. We see substantial deviation of the ϑ from the initial value ϑ_0.

In the second case the initial conditions are

$$s_0 = s_* = 0.9,\ l_0 = \frac{\pi}{2},\ d_0 = (1 - a_0)s_0,\ \delta_0 = 0, \tag{61}$$

The angle ϑ changes slightly now (fig. 5.b), but we note high frequency oscillation of ϑ that may cause high angular accelerations of the gyrostat.

5.3. Controlled motion of the prolate gyrostat with axisymmetric rotor (the cases 3 and 4)

Now let us examine the behavior of the gyrostat with internal torque

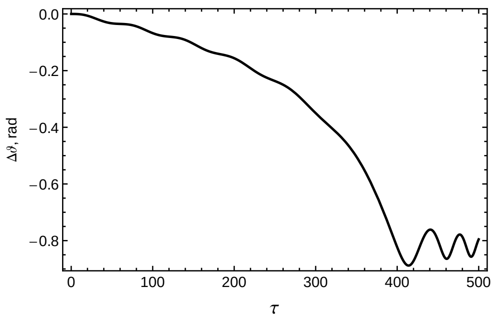
$$g_\delta = g_\delta^{pr} = \frac{C_p A_r'(\tau)}{[A_p + A_r(\tau)]^2} s_* \tag{62}$$

Fig. 5.c shows how the angle $\Delta\theta$ changes with the dimensionless time for the gyrostat that starts with the initial conditions (60). We note that the control torque (62) keeps initial angle between the axis $\boldsymbol{a}$ and the total angular momentum $\mathbf{G}$. For the initial condition (61) the gyrostat also keeps the angle ϑ (fig. 5.d).

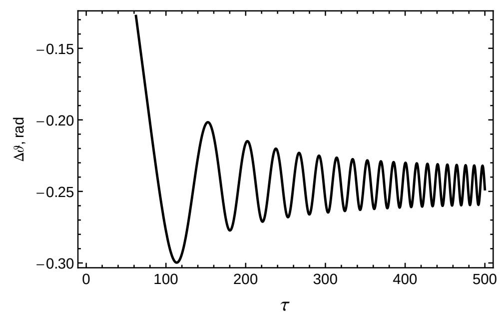
5.4. Controlled motion of the gyrostat width small asymmetry rotor (cases 5,6)

Let us examine the behavior of the gyrostat with small asymmetry rotor described by the equations (44)-(47) and with internal control (62). Fig. 5.e shows the change of the angle $\Delta\theta$ with dimensionless time for the initial condition (60) and the asymmetry $\varepsilon = 0.01$. We see that the control torque (62) doesn't preserve initial angle between the axis z_p of the gyrostat and the total angular momentum $\mathbf{G}$. Deviation of the angle θ from the initial condition is several orders greater than the angle deviation for the cases 3 and 4. In much the same way behaves the gyrostat with the initial condition (61) fig. 5.f.

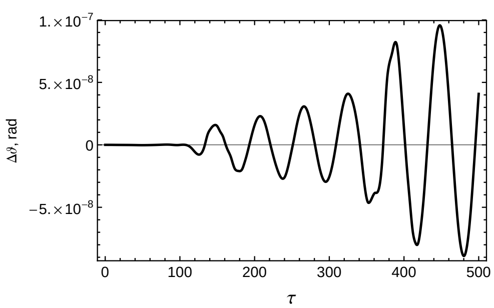
a) case 1: $\varepsilon = 0.00$, $\cos\vartheta_0 = 0.5$, $g_a = 0$



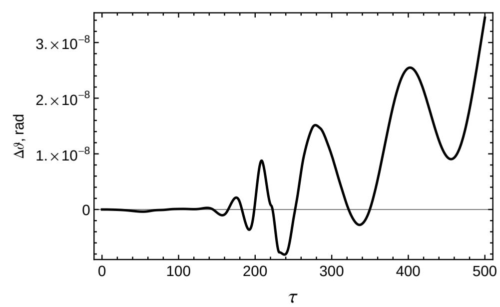
b) case 2: $\varepsilon = 0.00$, $\cos\vartheta_0 = 0.9$, $g_a = 0$



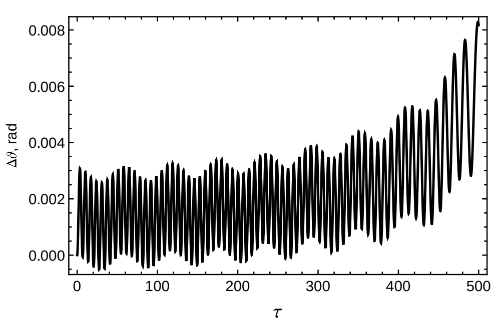
c) case 3: $\varepsilon = 0.00$, $\cos\vartheta_0 = 0.5$, $g_a \neq 0$



d) case 4: $\varepsilon = 0.00$, $\cos\vartheta_0 = 0.9$, $g_a \neq 0$



e) case 5: $\varepsilon = 0.01$, $\cos\vartheta_0 = 0.5$, $g_a \neq 0$



f) case 6: $\varepsilon = 0.01$, $\cos\vartheta_0 = 0.9$, $g_a \neq 0$

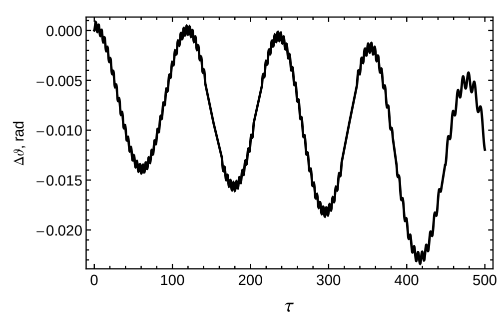


Figure 5: The deviation of the angle ϑ from its initial value $\Delta\vartheta = \vartheta - \vartheta_0$ for 6 cases

Conclusion

The dynamics of the dual-spin gyrostat spacecraft is described by using ordinary differential equations in the Serret-Andoyer canonical variables. The equations of motion are transformed into a simple dimensionless form with a small parameter. The stationary solutions are found for the undisturbed motion of the gyrostat. The control law obtained on the basis of the stationary solutions for the disturbed motion of the gyrostat with time-dependent moments of inertia. Several numerical examples are given to confirm the effectiveness of the control. It's shown that uncontrolled gyrostat spacecraft can lose its axis orientation. Under the action of the internal torque the angle between the axis of the gyrostat and the angular momentum vector is preserved. Internal torque g_a is constructed for the certain gyrostat type (oblate, prolate or intermediate) with axisymmetric rotor. The stationary solutions on the basis of which the control law is obtained are not the stationary solutions of the equations for the gyrostat with a asymmetric rotor. Therefore the proposed control is not suitable for such gyrostats.

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