Chaotic dynamics of an unbalanced gyrostat

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ABSTRACT

The free three-dimensional motion of an unbalanced gyrostat about the centre of mass is considered. The perturbed Hamiltonian for the case of small dynamical asymmetry of the rotor is written in Andoyer–Deprit canonical variables. The structure of the phase space of the unperturbed system is analysed, six forms of possible phase portraits are identified, and the equations of the phase trajectories are found analytically. Explicit analytical time dependences of the Andoyer–Deprit variables corresponding to heteroclinic orbits are obtained for all the phase portrait forms. The Melnikov function of the perturbed system is written for heteroclinic separatrix orbits using the analytical solutions obtained, and the presence of simple zeros is shown numerically. This provides evidence of intersections of the stable and unstable manifolds of the hyperbolic points and chaoticization of the motion. Illustrations of chaotic modes of motion of the unbalanced gyrostat are presented using Poincaré sections.

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Recently, much attention has been focused on the problems of the complex dynamics and chaoticization of the modes of motion of a gyrostat, which are associated with the variability of the composition (mass), as well as with the occurrence of external and internal perturbations of various kinds. The modes of motion that appear when the inertia–mass parameters vary with time and under the action of gravitational forces and perturbations associated with internal factors, such as the presence of cavities filled with a liquid within the gyrostat and a small degree of asymmetry of the rotor body, should be pointed out. The main method used in published studies to detect the appearance of chaos analytically has been the Melnikov–Wiggins formalism, based on finding simple zeros of the Melnikov function that are evidence of intersections of stable and unstable manifolds and separatrix splitting. In a series of studies Marsden–Holmes methods were used to study chaotic behaviour, the fundamental non-integrability of the equations of motion was demonstrated using Kozlov's method, and the fact that there is no additional analytic first integral was proved. The motion of gyrostats is generally described using the components of the angular momentum, as well as Andoyer–Deprit canonical variables. It is important to note that in the publications cited, the analytical dependences for the unperturbed separatrices in the space of Andoyer–Deprit variables, which are needed to write the Melnikov function, were constructed using known analytical dependences for homoclinic solutions in the three-dimensional space of the components of the angular momentum of a rigid body and a balanced gyrostat. The change to angles and Andoyer–Deprit momenta is usually made using relations of the kinematic type without integrating the canonical equations.

Unlike to the existing studies, in this paper analytical solutions for the heteroclinic orbits of a balanced gyrostat are found by directly integrating the canonical equations in Andoyer–Deprit variables, while taking into account the bifurcation changes in the structure of the phase space.

1. Statement of the problem

We will consider the free three-dimensional motion about the centre of mass of an unbalanced gyrostat formed by two bodies with triaxial ellipsoids of inertia. We introduce the following systems of coordinates (Fig. 1): OXYZ — the inertial system of coordinates, Ox1y1z1 — the connected principal system of coordinates of the carrier body, and Ox2y2z2 — the connected principal system of coordinates of the rotor body. The OZ1 and OZ2 axes of the connected systems are identical to the common rotation axis of the coaxial bodies. The angular velocity vectors of the bodies are represented in projections onto the axes of the Ox1y1z1 and Ox2y2z2 connected systems of coordinates by the relations $\omega_1 = (p', q', r')^T$ and $\omega_2 = (p, q, r)^T$. Here the projections of the angular velocity of the rotor are expressed in terms of the
angular velocity of the carrier body as follows:

\[ p' = p \cos \delta + q \sin \delta, \quad q' = q \cos \delta - p \sin \delta, \quad r' = r + \sigma \quad (\sigma = \dot{\delta}) \]  

(1.1)

where \( \delta \) is the relative angle of twist of the rotor body (the angle between the \( Ox_2 \) and \( Ox_1 \) axes). The expressions for the kinetic energy and the angular momentum, projected onto the axes of the \( Ox_2y_2z_2 \) system of coordinates, have the form

\[ T = \frac{1}{2} \left( A_1 (p \cos \delta + q \sin \delta)^2 + B_1 (q \cos \delta - p \sin \delta)^2 + C_1 (r + \sigma)^2 + A_2 p^2 + B_2 q^2 + C_2 r^2 \right) \]

(1.2)

\[ K = \left[ p (A_1 \cos^2 \delta + B_1 \sin^2 \delta + A_2) + q (A_1 - B_1) \sin \delta \cos \delta \right] \mathbf{i} + \left[ q (A_1 - B_1) \sin \delta \cos \delta + q (A_1 \sin^2 \delta + B_1 \cos^2 \delta + B_2) \right] \mathbf{j} + \left[ (C_1 + C_2) r + C_1 \right] \mathbf{k} \]

(1.3)

where \( A_i, B_i \) and \( C_i \) are the principal moments of inertia of the bodies (\( i = 1, 2 \)). To be specific, suppose \( A_1 > B_1 \) and \( A_2 > B_2 \). We introduce the dimensionless parameter

\[ \varepsilon = (A_1 - B_1)/A_1 > 0 \]

(1.4)

that characterizes the dynamical asymmetry of the rotor body. Note that the description of the motion in which \( A_2 < B_2 \) and (or) \( A_1 < B_1 \) can be reduced to the stated case without loss of generality by re-designating the axes of the connected systems of coordinates and adjusting the initial conditions.

We will now describe the dynamics of the system in Andoyer–Deprit variables. In these variables the position of the main carrier body is specified by three angles: \( \varphi_3, \varphi_2 \) and \( l \), which characterize rotations about the \( OZ \) axis, the direction of the angular momentum of the system, and the direction of the \( OZ_2 \) axis, respectively. The expressions for the generalized Andoyer–Deprit momenta are written, by definition, as follows:

\[ L = \frac{\partial T}{\partial l} = K \cdot \mathbf{k}, \quad I_2 = \frac{\partial T}{\partial \varphi_2} = K \cdot \mathbf{s} = K, \quad I_3 = \frac{\partial T}{\partial \varphi_3} = K \cdot \mathbf{k'}, \quad \Delta = \frac{\partial T}{\partial \delta} = C_1 (r + \sigma) \]

Note that the generalized momenta \( I_2 \) and \( I_3 \) are projections of the angular momentum of the system onto the \( OZ_2 \) and \( OZ \) axes and that \( I_2 \) is equal to the magnitude of the angular momentum vector; therefore, \( L \leq I_2 \). As is well known, the components of the angular momentum can be expressed in Andoyer–Deprit variables as follows:

\[ K_{x_2} = \sqrt{I_2^2 - L^2} \sin l, \quad K_{y_2} = \sqrt{I_2^2 - L^2} \cos l, \quad K_{z_2} = L \]

(1.5)
Comparing the components of the angular momentum of the system, we can use expressions (1.5) and (1.3) to express projections of the angular velocities in terms of Andoyer–Deprit variables

\[ p = \sqrt{I_1^2 - \frac{L^2}{S}} \left[ Q \sin l - (A_1 - B_1) \sin \delta \cos \cos l \right] \]
\[ q = \sqrt{I_1^2 - \frac{L^2}{S}} \left[ R \cos l - (A_1 - B_1) \sin \delta \cos \sin l \right] \]
\[ r = \frac{L - \Delta}{C_2}, \quad \sigma = \frac{\Delta - L - \Delta}{C_1} \]

(1.6)

where

\[ S = (A_1 + B_2)(B_1 + A_2) \sin^2 \delta + (A_1 + A_2)(B_1 + B_2) \cos^2 \delta \]
\[ Q = A_1 \sin^2 \delta + B_2 \cos^2 \delta + B_2, \quad R = A_1 \cos^2 \delta + B_2 \sin^2 \delta + A_2 \]

When relation (1.4) is taken into account, equatorial angular velocities (1.6) can be rewritten in a form that contains the dimensionless parameter \( \epsilon \):

\[ p = \sqrt{I_1^2 - \frac{L^2}{S}} \left[ (A_1 + B_2) \sin l - \epsilon A_1 \cos \delta \cos (l + \delta) \right] \]
\[ q = \sqrt{I_1^2 - \frac{L^2}{S}} \left[ (A_1 + A_2) \cos l - \epsilon A_1 \sin \delta \sin (l + \delta) \right] \]

where

\[ S = G - \epsilon A_1 E(\delta), \quad G = (A_1 + B_2)(A_1 + A_2), \quad E(\delta) = A_1 + A_2 \cos^2 \delta + B_2 \sin^2 \delta \]

Substituting expressions (1.6) into relation (1.2), we write the Hamiltonian of the system in Andoyer–Deprit variables. If the dimensionless parameter \( \epsilon \) is small, we can separate the expression for the Hamiltonian into an unperturbed part \( H_0 \) and a perturbed part \( \epsilon H_1 \), which is proportional to the first order of \( \epsilon \):

\[ H = T = H_0(l, l, I_2, L_1, L_2) + \epsilon H_1(l, \delta, L, I_2) \]

(1.7)

\[ H_0 = \frac{I_1^2 - \frac{L^2}{2}}{2} \left[ \frac{\Delta^2}{C_1} + \frac{(L - \Delta)^2}{C_2} \right] \]

(1.8)

\[ H_1 = \frac{A_1^2 (I_1^2 - \frac{L^2}{2})}{G} \]

\[ \times \left\{ E(\delta) \xi(l) - \sin^2 (l + \delta) - \frac{1}{2G} [(A_1 + B_2) \sin \sin \delta - (A_1 + A_2) \cos \cos \delta]^2 \right\} \]

(1.9)

\[ \xi(l) = \frac{\sin^2 l + \cos^2 l}{A_1 + A_2} + \frac{\cos^2 l}{A_1 + B_2} \]

(1.10)

The canonical system of Hamilton equations in Andoyer–Deprit variables takes the form

\[ \dot{q}_1 = \frac{\partial H_0}{\partial p_1} + \epsilon \frac{\partial H_1}{\partial p_1}, \quad \dot{p}_1 = -\frac{\partial H_0}{\partial q_1} - \epsilon \frac{\partial H_1}{\partial q_1} \]

(1.11)

where \( q_1 = (l, \varphi_2, \varphi_3, \Delta) \) and \( p_1 = (L, I_2, I_3, \Delta) \) are the canonical coordinates and momenta.

Taking into account expressions (1.8) and (1.9), we write Eqs (1.11) in the form

\[ \dot{l} = \frac{\partial H_0}{\partial L} + \epsilon \frac{\partial H_1}{\partial L}, \quad \dot{\varphi}_2 = \frac{\partial H_0}{\partial I_2} + \epsilon \frac{\partial H_1}{\partial I_2}, \quad \dot{\varphi}_3 = 0, \quad \dot{\Delta} = \frac{\partial H_0}{\partial \Delta} \]

\[ \dot{L} = -\frac{\partial H_0}{\partial l} - \epsilon \frac{\partial H_1}{\partial l}, \quad \dot{I}_2 = 0, \quad \dot{I}_3 = 0, \quad \dot{\Delta} = -\epsilon \frac{\partial H_1}{\partial \delta} \]

(1.12)

2. Structure of the phase space of the unperturbed system

In the unperturbed case (\( \epsilon = 0 \)), when the rotor has axial dynamical symmetry (\( A_1 = B_1 \)), Hamiltonian (1.8) can be rewritten, apart from a constant (\( \Delta = \text{const} \)), in the form

\[ H_0(l, L) = g(L) + a_1 (I_1^2 - \frac{L^2}{S}) \cos 2l = h = \text{const} \]

(2.1)
Hamiltonian (2.1) corresponds to the unperturbed system of equations

\begin{align*}
\dot{L} &= -\frac{\partial H}{\partial \dot{L}} = \frac{L}{C_2} - \frac{\Delta}{C_2} - 2L(a_1 + a_2 \cos 2l), \\
L &= -\frac{\partial H}{\partial \dot{L}} = 2a_2(I_2^2 - L^2) \sin 2l
\end{align*}

(2.3)

The phase space of system (2.3) is the involute of the following cylinder (ring):

\[ U = \left\{ (l, L) \in \mathbb{R}^2 : \quad (l, L) \in [-1, 1] \right\} \]

(2.4)

By virtue of Eqs (2.3), the following groups of special points \((l, L)\) that correspond to equilibrium positions \((\dot{l}, \dot{L}) = (0, 0)\), are possible:

1) \(l_* = \pi n, \quad L_* = \frac{\Delta}{C_2} \left( \frac{1}{C_2} - 2[a_1 + a_2] \right) \frac{1}{a_1 + B_2 - C_2} \Delta \)

2) \(l_* = \frac{\pi}{2} + \pi n, \quad L_* = \frac{\Delta}{C_2} \left( \frac{1}{C_2} - 2[a_1 - a_2] \right) \frac{1}{a_1 + A_2 - C_2} \Delta \)

3) \(L_* = I_2, \quad \cos 2l_* = \frac{1}{a_2} \left( \frac{I_2 - \Delta}{2I_2 C_2} - a_1 \right) \)

4) \(L_* = -I_2, \quad \cos 2l_* = \frac{1}{a_2} \left( \frac{I_2 + \Delta}{2I_2 C_2} - a_1 \right) \)

(2.5) \quad (2.6) \quad (2.7) \quad (2.8)

Note that, generally speaking, by virtue of the second equality in (2.3), the phase portrait includes special points with the coordinate \(l = \pi k/2\), which, however, must be separated into two different groups: when \(k\) is an even number \((k = 2n, \ l = \pi n)\), points (2.5) appear, and when \(k\) is an odd number \((k = 1 + 2n, \ l = \pi (1 + 2n)/2 = \pi/2 + \pi n)\), points (2.6) appear. It is seen that the corresponding values of \(L\) are different for points (2.5) and (2.6).

An analysis of these special points shows that different types of phase portraits, whose structure is characterized by the bifurcation diagram shown in Fig. 2 \((D = A_1 - C_2)\), are possible for different combinations of the moments of inertia. To be specific, suppose \(D < 0\) and \(\Delta > 0\). The phase portrait of the system can be of one of six qualitative forms (Fig. 3).
Table 1 shows the correspondence between the qualitative forms and the constraints on the value of the momentum $\Delta$, which takes into account the differences between the moments of inertia in the different zones. The following auxiliary quantities have been introduced

$$\alpha = 1 - \frac{C_2}{A_1 + A_2} , \quad \beta = 1 - \frac{C_2}{A_1 + B_2} , \quad \gamma = \inf \{ |\alpha|, |\beta| \} , \quad \overline{\gamma} = \sup \{ |\alpha|, |\beta| \}$$

Note that if $D \geq 0$, we will have only zone 1, and if $\Delta < 0$, we will have a phase portrait that can be reflected symmetrically relative to the Ol coordinate axis. As is seen from Fig. 3, phase portrait qualitative forms 1 and 5 are similar to the pendulum type with two oscillatory regions (in the vicinity of points of the “centre” type), two rotational regions (upper and lower), and two separatrices. Forms 1 and 5 are often encountered in rigid body dynamics, and they, unlike the other forms, may be characterized as traditional forms. Forms 2 and 4...
Table 1

<table>
<thead>
<tr>
<th>Zone or boundary</th>
<th>Critical value $\Delta^*$</th>
<th>Constraints for $\hat{\Delta} = \Delta/\delta$</th>
<th>Phase portrait qualitative form (Fig. 3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zone 1 ($\alpha &gt; \beta &gt; 0$)</td>
<td>$\alpha \delta$</td>
<td>$0 \leq \hat{\Delta} \leq \beta$</td>
<td>1</td>
</tr>
<tr>
<td>Boundary 1 ($\alpha &gt; 0, \beta &gt; 0$)</td>
<td>$\alpha \delta$</td>
<td>$0 \leq \hat{\Delta} \leq \alpha$</td>
<td>2</td>
</tr>
<tr>
<td>Zone 2 ($\alpha &gt; 0, \beta &lt; 0$)</td>
<td>$\gamma \delta$</td>
<td>$\gamma &lt; \hat{\Delta} \leq 0$</td>
<td>3</td>
</tr>
<tr>
<td>Boundary 2</td>
<td>$-\beta \delta$</td>
<td>$0 \leq \hat{\Delta} \leq -\beta$</td>
<td>4</td>
</tr>
<tr>
<td>Zone 3 ($0 &gt; \alpha &gt; \beta$)</td>
<td>$-\beta \delta$</td>
<td>$0 \leq \hat{\Delta} \leq -\alpha$</td>
<td>5</td>
</tr>
</tbody>
</table>

Table 2

<table>
<thead>
<tr>
<th>Zone or boundary</th>
<th>Phase portrait qualitative form</th>
<th>Solution case</th>
<th>Saddle points</th>
<th>Value of constant (3.2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zone 1</td>
<td>I</td>
<td>(2.5)</td>
<td>$h^+_s$</td>
<td></td>
</tr>
<tr>
<td>Boundary 1</td>
<td>II</td>
<td>(2.7)</td>
<td>$h^+_1$, $h^-_1$</td>
<td></td>
</tr>
<tr>
<td>Zone 2</td>
<td>III</td>
<td>(2.7)</td>
<td>$h^+_2$, $h^-_2$</td>
<td></td>
</tr>
<tr>
<td>Boundary 2</td>
<td>IV</td>
<td>(2.7)</td>
<td>$h^+_3$, $h^-_3$</td>
<td></td>
</tr>
<tr>
<td>Zone 3</td>
<td>V</td>
<td>(2.6)</td>
<td>$h^+_4$, $h^-_4$</td>
<td></td>
</tr>
<tr>
<td>Any zone</td>
<td>No special points</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Each column in Table 2 contains two oscillatory regions and one rotational region, i.e., the lower or upper, respectively. Form 3 has four oscillatory regions and one “middle” rotational region. Form 6 consists entirely of a single rotational region. Thus, there are six qualitative forms of phase portraits, depending on the value of the momentum $\Delta$. Regardless of the zone or boundary, when the momentum $\Delta$ exceeds the critical value $\Delta^*$ (Table 1), the phase portrait takes on form 6 (the “rotational” form), which does not contain any special point from the four groups.

It is important to note that the equation of an arbitrary phase trajectory of an unperturbed system with an appropriate value of the Hamiltonian $h$ can be written using expression (2.1) in the form

$$l(L) = \pm \frac{1}{2} \arccos \frac{h - g(L)}{a_2(I_2^2 - L^2)}$$

(2.10)

3. Heteroclinic trajectories

To detect the appearance of chaos in a perturbed system analytically, we use Wiggins’ method to calculate the Melnikov function.

For this purpose, we find analytical solutions of the equations of unperturbed motion (2.3) in explicit form for the heteroclinic trajectories corresponding to the unperturbed separatrices.

The separatrices that pass through saddle points separate the oscillatory and rotational regions of the phase space, form heteroclinic orbits and are specified by Eq. (2.10) with the value of the Hamiltonian calculated at the saddle point:

$$h = h_s = H_0(l_s, L_s)$$

(3.1)

Depending on the zone and the qualitative form, the saddle points connected by heteroclinic trajectories can be points of all four types (2.5)-(2.8). The constants $h_s$ depend on the form of the phase portrait and the zone, and the following values can be used (Table 2)

$$h^+_s = g^+_s(L_\delta) = (a_1 \pm a_2)\left(I_2^2 - L_s^2\right) + \frac{L_\Delta}{2C_2} - \frac{L_\delta}{C_2}, \quad h^-_s = \frac{I_x^2}{2C_2} + \frac{L_\delta}{C_2}$$

(3.2)

Table 2 shows all possible combinations of zones and qualitative forms.

Using relation (2.10), we eliminate the $l$ coordinate in the second equality in (2.3) and reduce it to a form containing a difference between squares under the radical:

$$\hat{L} = 2\sqrt{[a_1(I_2^2 - L^2)]^2 - [h_s - g(L)]^2} = 2\sqrt{-f_4(L)f_5(L)}$$

(3.3)

$$f_4(L) = g_4(L) - h_s$$

(3.4)

Next, we will find the heteroclinic separatrices solutions in all the cases indicated by Roman numerals in Table 2.

Case I. We first consider qualitative forms 1 in zone 1 and qualitative form 5 in zone 3 (Table 2), i.e., the “traditional” forms of the phase space of a free rigid body and a balanced gyrostat. Form 1, the quadratic polynomial $f_4(L)$ has the multiple root $L = L_s$; here and below, $L_s$ refers to the value of $L$ in the corresponding case of the saddle point and the constant $h_s$ (see Table 2). The polynomial $f_4(L)$ has
two different roots $L_1$ and $L_2$ ($L_2 < L_1$), which are easily found:

$$L_{1,2} = \frac{-\Delta \pm \sqrt{\Delta^2 - 4\Delta}}{2\Delta C_2(a_1 - a_2) - 1}$$  \hspace{1cm} (3.5)$$

For form 5, the polynomial $f(L)$ has the multiple root $L = L_s$, and the polynomial $f_2(L)$ has two different roots:

$$L_{1,2} = \frac{-\Delta \pm \sqrt{\Delta^2 - 4\Delta}}{2\Delta C_2(a_1 + a_2) - 1}$$  \hspace{1cm} (3.6)$$

Here and in relation (3.5) we have

$$d_\pm = \left[1 - 4a_2 C_2 + 4C_2^2 \left(a_1^2 - a_2^2\right)\right]L_s^2 + 2\Delta \left[2C_2(a_1 \pm a_2) - 1\right]L_s + \Delta^2 + 4C_2^2 a_2^2 \left[2C_2(a_1 \pm a_2) \mp 1\right]$$

Note that the roots $L_1$ and $L_2$ ($L_2 < L_1$) correspond to the “extremum” points of the separatrices (Fig. 3, form 5). When the roots found are taken into account, Eq. (3.3) can be rewritten in the form

$$L = 2\sqrt{-f(L)f_2(L)} = 2\sqrt{-k(L - L_s)^2(L - L_1)(L - L_2)}$$  \hspace{1cm} (3.7)$$

where

$$k = \left(\frac{1}{2C_2} - (a_1 + a_2)\right)\left(\frac{1}{2C_2} - (a_1 - a_2)\right) = \frac{\alpha \beta}{4C_2^2},$$  \hspace{1cm} (3.8)$$

(the parameters $\alpha$ and $\beta$ were introduced by the first two equalities in (2.9)).

In zones 1 and 3, in which phase portrait qualitative forms 1 and 5 appear, $k > 0$. The range of variation of the momentum $L \in (L_1, L_s)$ corresponds to motion on the upper separatrix, and the range $L \in (L_2, L_s)$ corresponds to motion on the lower separatrix.

We separate the variables and transform Eq. (3.7) into the following integral

$$2\sqrt{k} = \int \frac{L - L_1}{\sqrt{(L - L_s)(L_1 - L)(L - L_2)}}$$  \hspace{1cm} (3.9)$$

During the further transformation of expression (3.9), we should take into account the directions of motion on the separatrices, the differences between the signs of the quantity $(L - L_s)$ on the upper and lower separatrices, and the symmetrical picture of the motion on the separatrices, beginning from the “extremum” points in forward and reverse time. If we use the replacement $x = L - L_s$ and take the last remarks into account, which enable us to determine the sign, integral (3.9) reduces to the known expression

$$2\sqrt{k} = \int \frac{dx}{x \sqrt{R(x)}} = \frac{1}{\sqrt{a}} \ln \frac{2a + bx + 2\sqrt{aR(x)}}{x} + C$$  \hspace{1cm} (3.10)$$

where

$$R(x) = a + bx + cx^2, \quad a = L_s(L_1 + L_2) - L_1 L_2 - L_s^2, \quad b = L_1 + L_2 - 2L_s, \quad c = -1$$  \hspace{1cm} (3.11)$$

Quadrature (3.10) is valid only when $a > 0$.

It can be shown that the value of $a$ calculated in the case of qualitative form 1 ($\alpha > 0$ in zone 1) is positive:

$$a = 4\alpha a_2 C_2 \left(L_s^2 - L_1^2\right) > 0$$

In the case of qualitative form 5 ($\beta < 0$ in zone 3), the value of $a$ is also positive:

$$a = \frac{4}{\beta} a_2 C_2 \left(L_s^2 - L_1^2\right) > 0$$

The inverse replacement $L = x + L_s$ in equality (3.10) gives the general solution for the momentum in the unperturbed system $L(t)$, which corresponds to the separatrices for phase portrait qualitative forms 1 and 5:

$$L(t) = L_s + \frac{4\sqrt{a} \exp(\lambda t)}{[\sqrt{a} \exp(\lambda t) - b]^2 - 4ac}; \quad \lambda = \lambda^{(i)} = 2\sqrt{ak}$$  \hspace{1cm} (3.12)$$

The integration constant $E$ in solution (3.12) is determined from the condition of passage through the “extremum” point at the initial time. We then have $E = E_1^{(1)} = \frac{b + 2a(L_1 - L_s)}{2}$ for the upper separatrix and $E = E_2^{(1)} = \frac{b + 2a(L_2 - L_s)}{2}$ for the lower separatrix.

The explicit time dependence of the coordinate $l(t)$ for the unperturbed separatrix can be obtained by substituting solution (3.12) into expression (2.9).
Case II. We will now obtain the solution for the separatrix in the case of phase portrait qualitative form 2 for zone 1 (Table 2, Fig. 2, form 2). Here the saddle points become points of the third type (2.7). The separatrices $S_1S_2$ and $S_3S_4$, which pass through these saddle points, are characterized by $h_i$ (3.2). In this case both polynomials in (3.4) have two different roots, one of which is

$$L = L_3 = I_2$$  \tag{3.13}$$

The second root of the polynomial $f_s(L)$ equals

$$L_1 = 2\Delta/\beta - I_2$$  \tag{3.14}$$

and the second root of the polynomial $f_a(L)$ has the value

$$L_2 = 2\Delta/\alpha - I_2$$  \tag{3.15}$$

When these roots are taken into account, differential equation (3.7) can be rewritten in the form (3.9) and integrated in the form (3.10), where we now have

- $a = r_1 r_2$, $b = r_2 - r_1$, $c = -1$;
- $r_1 = I_2 - L_1 = 2(I_2 - \Delta/\beta)$,
- $r_2 = L_2 - I_2 = 2(\Delta/\alpha - I_2)$  \tag{3.16}$$

It follows from the constraints imposed on the value of $\Delta/L_2$, which are indicated in Table 1 that for qualitative form 2 in zone 1 the parameters have values $r_1 < 0$ and $r_2 < 0$; therefore, the positivity requirement of the coefficient $a$ for quadrature (3.10) holds, and solution (3.12) remains valid. It should be noted that the heteroclinic orbit $S_3S_4$ is distinguished from the orbit $S_1S_2$ only by displacement along the $l$ coordinate by $\pi$ and the single integration constant has the form

$$E = E^{(2)} = \frac{2\Delta C_2 B_2 - A_2}{(A_1 + A_2 - C_2)(A_1 + B_2 - C_2)}$$  \tag{3.17}$$

Case III. Consider phase portrait qualitative form 2 in zone 2 (Table 2), In zone 2 the value of $k$ (3.8) is negative. For qualitative form 2 the separatrices correspond to the constant $h_i^-$ (3.2). In this case, as in the preceding case, polynomials (3.4) have roots (3.13), (3.14) and (3.15). By virtue of the last remarks, integral (3.9) should be rewritten in a form similar to (3.9) (after replacing $k$ and $L_1 - L$ by $-l - L_1$). After the replacement of variables $x = L - I_2$ and isolation of the quadratic polynomial under the radical, the integral is transformed into a quadrature and again enables us to write the solution $\bar{L}(t)$ in the form (3.12) for the parameters

$$k < 0, \quad a = r_1 r_2 > 0, \quad b = -r_2 - r_1, \quad c = 1$$

$$r_1 = L_1 - I_2 = 2(\Delta/\beta - I_2) < 0, \quad r_2 = L_2 - I_2 = 2(\Delta/\alpha - I_2) < 0$$

$$\lambda = 2\sqrt{-ak}, \quad E = -E^{(2)}$$  \tag{3.18}$$

Case IV. Consider phase portrait qualitative form 3 in zone 2 (Table 2), which contains four separatrices, namely, the upper pair $S_1S_2$ and $S_3S_4$, which pass through saddle (2.7) with the constant $h_i^-$ (3.2), and the lower pair $S_5S_6$ and $S_7S_8$, which pass through saddle (2.8) with the constant $h_i^- (3.2)$. The paired separatrices are distinguished from one another by a displacement along the $l$ coordinate by $\pi$; therefore, it is sufficient to examine one upper separatix and one lower separatix.

The solution for the upper separatix is identical to the preceding case, i.e., solution (3.12) with parameters (3.18) remains valid. We will find the solution for the lower separatix. Polynomials (3.4) with the constant $h_i^-$ will have two different roots, which, as in case II, are specified by formulae (3.13)-(3.15), but with $l_2$ replaced by $-l_2$.

Performing calculations similar to the preceding cases, we can obtain the solution $\bar{L}(t)$ in the form (3.12) with the same parameters (3.18), but with $I_2$ replaced by $-I_2$ and $E = -E^{(2)}$ (now we have $r_1 > 0$, $r_2 > 0$).

The solution for the upper separatix is identical to the preceding case, i.e., solution (3.12) with parameters (3.18) remains valid. We will find the solution for the lower separatix. Polynomials (3.4) with the constant $h_i^-$ will have two different roots, which, as in case II, are specified by formulae (3.13)-(3.15), but with $l_2$ replaced by $-l_2$.

Performing calculations similar to the preceding cases, we can obtain the solution $\bar{L}(t)$ in the form (3.12) with the same parameters (3.18), but with $l_2$ replaced by $-l_2$ and $E = -E^{(2)}$ (now we have $r_1 > 0$, $r_2 > 0$).

Case V. Consider qualitative form 4 in zone 3 (Table 2), which contains the pair of separatrices $S_1S_2$ and $S_3S_4$ (which are displaced along the $l$ coordinate by $\pi$), which pass through saddle (2.8) with the constant $h_i^-$ (3.2). In zone 3 the parameter $k$ (3.8) is positive. Polynomials (3.4) have the same roots as in case IV (qualitative form 3 in zone 2, the lower separatix). Repeating transformations similar to the previous cases, we obtain the solution $\bar{L}(t)$ in the form (3.12) with the parameters

$$k > 0, \quad a = r_1 r_2 > 0, \quad b = r_2 - r_1, \quad c = -1$$

$$r_1 = -2(\Delta/\beta + I_2) < 0, \quad r_2 = 2(\Delta/\alpha + I_2) < 0$$

$$\lambda = 2\sqrt{ak}, \quad E = -E^{(2)}$$  \tag{3.19}$$

Case VI. Consider phase portrait qualitative form 2 on boundary 1 (Table 2) when there is a pair of separatrices that pass through saddle (2.7) with the constant $h_i^-$ (3.2). Since $\beta = 0$, $A_1 + B_2 - C_2 = 0$ in the case under consideration, polynomials (3.4) take the form

$$f_s(L) = \frac{\Delta}{C_2} (I_2 - L), \quad f_a(L) = \frac{\Delta}{C_2} \left( \alpha \frac{I_2 + L}{2C_2} \right) (I_2 - L)$$

Then, after the replacement of variables $x = L - I_2$, Eq. (3.3) is transformed into the integral

$$2t = \int_0^\infty \frac{dx}{x\sqrt{ax + b}}, \quad a = -\frac{\Delta}{2C_2}, \quad b = \frac{\Delta}{C_2} (\chi I_2 - \Delta), \quad \chi = \frac{A_2 - B_2}{A_1 + A_2}$$  \tag{3.20}$$
When we impose the existing constraints (Table 1) and take into account the fact that the equality $A_1 + B_2 = C_2$ holds on boundary 1, we can show the positivity of $b$:

$$\chi = \frac{A_2 - B_2}{A_1 + A_2} = 1 - \frac{C_2}{A_1 + A_2} = \alpha, \quad b = \frac{A_2}{C_2}(\alpha I_2 - \Delta) > 0$$

Then integration leads to the following result:\textsuperscript{18}

$$2r = \frac{1}{\sqrt{b}} \ln \frac{\sqrt{ax + b} - \sqrt{b}}{\sqrt{ax + b} + \sqrt{b}} - \ln C, \quad C = \text{const}$$

After the inverse replacement of variables and some reduction, from this equation we obtain the explicit dependence

$$\bar{L}(t) = \frac{b}{a} \left[ \frac{\exp(2\sqrt{b}t) + 1}{\exp(2\sqrt{b}t) - 1} \right]^2 - \frac{b}{a} + I_2, \quad E = -1$$

(3.22)

The value of the integration constant $E = -1$ follows from the condition of passage through the “extremum” point on the separatrix at the time $t = 0$.

Case VII. We consider phase portrait qualitative form 4 on boundary 2 (Table 2) when there is a pair of separatrices that pass through saddle (2.8) with the constant $\tilde{h}_2$ (3.2). In this case, the equalities

$$A_1 + A_2 - C_2 = 0, \quad \alpha = 0, \quad \beta = (B_2 - A_2)/(A_1 + B_2)$$

hold on boundary 2, and polynomials (3.4) reduce to the form

$$f_s(L) = \frac{-\Delta}{C_2} \left( \frac{\beta (I_2 - L)}{2C_2} \right) (I_2 + L), \quad f_u(L) = \frac{-\Delta}{C_2} (I_2 + L)$$

(3.23)

After the replacement $x = I_2 + L$ and separation of variables, Eq. (3.3) is again transformed using polynomials (3.23) into integral (3.20), in which the constant parameters have the values

$$a = \frac{-\Delta}{2C_2} \beta, \quad b = \frac{-\Delta}{C_2} (-\beta I_2 + \Delta) > 0$$

(3.24)

In this case the explicit dependence $\tilde{L}(t)$ is distinguished from (3.22) by the replacement of $I_2$ by $-I_2$.

Thus, analytical expressions have been obtained for all the heteroclinic solutions of the separatrix trajectories in all the zones and for all the qualitative forms of the phase portraits. Two types of separatrix solutions, which are specified by Eqs (3.12) and (3.22) and can be parametrically separated into the seven cases described above, have been found.

Note that the solutions obtained were found directly in the space of variables $\{l, L\}$ without changing to Andoyer–Deprit angles and momenta from known analytical dependences for homoclinic solutions in the space of components of the angular momentum $\{K_{x2}, K_{y2}, K_{z2}\}$ (Refs 7 and 14) using relations (1.5).

4. Calculation of the Melnikov function and demonstration of chaotization

As is well known, in dynamical systems with perturbations, modes of chaotic behaviour can appear, and they are associated with significant complication of the structure of the phase space in the vicinity of the unperturbed separatrices. This complication of the structure of the phase space is associated with an increase in its dimension (compared with the unperturbed system), and it is also caused by the appearance of intersections of the stable and unstable manifolds of the hyperbolic points that form the separatrices. These manifold intersections can be determined using the Melnikov function by finding its simple zeros. It should also be noted that the Melnikov function provides a way to discover only local chaotic motion near a heteroclinic trajectory.

Consider a closed dynamical system consisting of four equations of system (1.12)

$$\dot{i} = \frac{\partial H_0(i, l, \Lambda, \delta)}{\partial l} + \varepsilon g_i(i, l, \Lambda, \delta), \quad \dot{\delta} = \Omega(i, l, \Lambda, \delta) + \varepsilon g_\delta(i, l, \Lambda, \delta)$$

(4.1)

$$\dot{l} = \frac{-\partial H_0(l, \Lambda, \delta)}{\partial \delta} + \varepsilon g_L(l, \Lambda, \delta), \quad \dot{\Lambda} = \varepsilon g_\Lambda(l, \Lambda, \delta)$$

where

$$g_i(i, l, \Lambda, \delta) = \frac{\partial H_1}{\partial l}, \quad g_\delta(i, l, \Lambda, \delta) = 0, \quad \Omega(i, l, \Lambda, \delta) = \frac{\partial H_0}{\partial \Lambda}$$

$$g_L(l, \Lambda, \delta) = \frac{\partial H_1}{\partial \delta}, \quad g_\Lambda(l, \Lambda, \delta) = \frac{\partial H_1}{\partial \delta}$$

It follows from Eqs (1.12) that $I_2, I_3$ and $\varphi_3$ are constants. From this and from the fact the coordinate $\varphi_2$ does not appear in system (4.1) it follows that the system is closed. The solution for the coordinate $\varphi_2$ can be obtained separately from system (1.12) after integrating system (4.1).
Using the Melnikov–Wiggins formalism, we can write the Melnikov function for four-dimensional system (4.1) in the form

\[ M(t_0) = \int_{-\infty}^{\infty} g_L \frac{\partial H_0}{\partial L} + g_L \frac{\partial H_0}{\partial L} + g_L H_0 \frac{\partial H_0}{\partial \Delta} (\overline{t}(t), \overline{L}(t), \overline{\Delta}(t)) dt - \]

\[ \frac{\partial H_0}{\partial \Delta} (t, L) \int_{-\infty}^{\infty} g_L (\overline{t}(t), \overline{L}(t), \overline{\Delta}(t) + \Delta(t)) dt \]

(4.2)

where \( \Delta = \text{const} \) is the solution for the momentum \( \Delta(t) \) of the parent unperturbed system \( (\varepsilon = 0) \) and the function \( \tilde{\delta}(\tau) \) represents the following solution along the separatrix:

\[ \tilde{\delta}(\tau) = \int 0(\overline{t}(\tau), \overline{L}(\tau), \overline{\Delta}) d\tau = \int \left( \Delta(t) - \Delta(t + t_0) \right) d\tau = \]

\[ = \frac{\Delta(C_1 + C_2)}{C_1 C_2} \frac{J(\tau)}{C_2} + \text{const} \]

(4.3)

where

\[ J(\tau) = \int \overline{L}(\tau) d\tau \]

(4.4)

Note that in the case under consideration the phase space section was introduced and is used by means of Poincaré mapping based on the condition \( \delta_0 = \delta \mod 2\pi, \delta_0 = \text{const} \) (the operation \( a \mod b \) denotes obtaining the remainder from the division of \( a \) by \( b \)).
Integral (4.4) in expression (4.3) is calculated analytically as a function of the form of the solution $\tilde{I}(\tau)$ and a combination of parameters. For solution (3.12) this integral is expressed in inverse hyperbolic and trigonometric functions ($a > 0$):

\begin{align*}
J(\tau) &= L_2 \tau - \frac{2a}{\lambda \sqrt{ac}} \operatorname{arcth} \frac{E \exp(\lambda \tau) - b}{2\sqrt{ac}}, \quad c > 0 \\
J(\tau) &= L_2 \tau + \frac{2a}{\lambda \sqrt{|a|}} \operatorname{arctg} \frac{E \exp(\lambda \tau) - b}{2\sqrt{|a|}}, \quad c < 0
\end{align*}

(4.5)
For solutions of the form (3.22), integral (4.4) takes the form

\[ J(t) = \frac{2b}{a} \left[ E \exp\left( 2b t \right) - 1 \right] \pm I_x t \]  

(4.6)

Using solutions (3.12), (3.22), (4.5) and (4.6), we can write an explicit expression for the Melnikov function (4.2) and calculate it on the separatrix under investigation in any of the three zones and for all the phase portrait qualitative forms. Here we will confine ourselves to an example of calculating the Melnikov function in one of the zones (zone 3). The numerical calculation demonstrates that the Melnikov function has zeros (see Fig. 4, where \( A_1 = 3, C_1 = 2, A_2 = 8, B_2 = 7, C_2 = 12 \text{ kg m}^2, \Delta = 0.1 \text{ kg m}^2/\text{s}, \delta_0 = 0 \)) and, therefore, that separatrix splitting and chaotization of the motion occurs, which is clearly exhibited on the Poincaré sections in the form of chaotic layers in the vicinity of the separatrices (Fig. 5, Table 3). In Fig. 4 curve 1 corresponds to the Melnikov function calculated on the upper separatrix, and curve 2 corresponds to the Melnikov function calculated on the lower separatrix. The Poincaré sections (Fig. 5) were constructed for the parameters indicated in Table 3 and the following shared initial values

\[ I_2 = 10, \ I_1 = 1, \ \varphi_2 = 1, \ \varphi_3 = 1, \ \delta_0 = 0 \]

The last fragment (Fig. 5f) characterizes the stabilizing action of \( \Delta \) and demonstrates the reduction of the chaotic zone compared with the fragment in Fig. 5e even when there is some increase in the value of the small parameter.

Note that the dynamical effects of separatrix splitting fundamentally preclude integrability of the equations of motion. For example, the non-integrability of the equations of motion of an asymmetrical gyrostat in the vicinity of \( \Delta = 0 \) has been shown using Kozlov’s method, which proves that there is no additional analytic first integral.

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